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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

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product is then the reciprocal of a polynomial in λ^2 (as in the case d even), and the corresponding inversion formula is

$$u = P(L)\left((R^*Ru) * f_{a,b}\right) ,$$

where P is a polynomial. We refer to [1] for details.

Unfortunately the method of spherical transforms sketched above seems to provide explicit inversion formulas for the d-geodesic Radon transform on X only when $q' = q$ or $q' = 1$ on the one hand (to get rid of ${}_3F_2$) *and* d even or $X = H^n(\mathbf{R})$ on the other hand. The only reachable results so far are thus the formulas already obtained in [1] for $H^n(\mathbf{R})$ and a new proof of the above Theorem 8. The method might however yield some new results in the wider class of Damek-Ricci spaces (or harmonic NA groups), where the dimension q can be an arbitrary integer.

6. Shifted Radon transforms, waves, and the amusing formula

On page 146 of [10], S. Helgason notes the "amusing formula"

$$LR^*Ru(x) = -\frac{\partial}{\partial\tau}R^*_{t(\tau)}Ru(x)\Big|_{\tau=1} \tag{12}$$

for the 2-geodesic Radon transform R on the hyperbolic space $X = H^3(\mathbf{R})$, where L is the Laplace-Beltrami operator of X and $x \in X$. In this formula, R^*_t is the generalized dual transform obtained by integrating over all 2-dimensional totally geodesic submanifolds *at distance* t from a point x, and $t = t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$. In [10], or [11], p. 55, equation (12) is indirectly obtained by bringing together two different inversion formulas for R.

In this section we study general shifted transforms, a concept going back to Radon himself [16] for the line transform in $\mathbf{R}^2$, and we use them to derive inversion formulas. They also provide solutions of wave-type equations; formula (12) can actually be seen as a wave equation at time $t = 0$. We shall give a direct proof of some generalized "amusing formulas", thus solving wave equations (called multitemporal when the time variable is multidimensional), and we use them to relate two different types of Radon inversion formulas (with or without shifts).

6.1 Shifts

As before, let $X = G/K$ and $Y = G/H$ be two homogeneous spaces, with K compact, and

$$Ru(g \cdot y_o) = \int_H u(gh \cdot x_o)\, dh$$

be the corresponding Radon transform of $u \in C_c(X)$.

Let $t \in G$ be a "*shift*", fixed at first. Replacing the origin $y_o = H$ in Y by the shifted origin $y_t = t \cdot y_o$, with stabilizer subgroup $H_t = tHt^{-1} \subset G$, we obtain the new identification $Y = G/H_t$, and a new incidence relation between X and Y. A point $x = g \cdot x_o \in X$ is now incident to $y \in Y$ if and only if there exists $\gamma \in G$ such that

$$x = \gamma \cdot x_o \quad \text{and} \quad y = \gamma \cdot y_t = \gamma t \cdot y_o\,,$$

i.e.

$$y = gkt \cdot y_o\,,$$

for some $k \in K$. The corresponding *shifted dual transform* of $v \in C(Y)$ is

$$R_t^* v(g \cdot x_o) = \int_K v(gkt \cdot y_o)\, dk\,.$$

REMARK. We now have two double fibrations

$$\begin{array}{ccc} Z = G/(K \cap H) & & \\ \downarrow & \searrow & \\ X = G/K & & Y = G/H\,, \end{array} \qquad \begin{array}{ccc} Z_t = G/(K \cap H_t) & & \\ \downarrow & \searrow & \\ X = G/K & & Y = G/H_t\,, \end{array}$$

and we are dealing with the Radon tranform R given by the first and the dual transform R_t^* given by the second. The transform R_t associated with the second diagram is

$$R_t u(g \cdot y_o) = \int_H u(ght^{-1} \cdot x_o)\, dh\,;$$

but, excepting the proof of Proposition 12, it will not be used in the sequel.

LEMMA 11. *Let $u \in C_c(X)$ and $g, t \in G$. Then*

$$(R_t^* Ru)(g \cdot x_o) = (Ru_g)(t \cdot y_o)\,,$$

where u_g is the K-invariant function on X defined by

$$u_g(x) = \int_K u(gk \cdot x)\, dk\,.$$

Proof. Immediate, since

$$(R_t^* Ru)(g \cdot x_o) = \int_{K \times H} u(gkth \cdot x_o)\, dk\, dh = \int_H u_g(th \cdot x_o)\, dh . \qquad \square$$

Before proceeding we mention the following extension of Proposition 3 to shifted transforms. This result will not be used in the sequel.

PROPOSITION 12. *Let G and H be unimodular, K compact, $X = G/K$ and $Y = G/H$. For any $u \in C_c(X)$ and $t \in G$ we have*

$$R_t^* Ru = u * S_t$$

(convolution on X). Here S_t is the K-invariant distribution on X defined by $S = R_t^* R\delta$, and δ is the Dirac distribution at the origin $x_o = K$ of X, i.e.

$$\langle S_t, u\rangle = R^* R_t u(x_o) = \int_{K \times H} u(kht^{-1} \cdot x_o)\, dk\, dh .$$

Proof. The proof of Proposition 3 can be repeated here, with $R^* R_t$ as the dual of $R_t^* R$. The claim can also be checked directly, writing, for $\varphi \in \mathcal{D}(X)$,

$$\langle R_t^* Ru, \varphi\rangle = \int_{G \times H} u(gth \cdot x_o)\varphi(g \cdot x_o)\, dg\, dh ,$$

and changing variables into $h' = h^{-1}$, $g' = gth$; the result follows easily, G and H being unimodular groups. Details are left to the reader. $\square$

6.2 RADON INVERSION BY SHIFTS

The elementary Lemma 11 can be used in the following way. Assume the transform R can be inverted at the origin for K-invariant functions on X, say

$$u(x_o) = \langle T_{(y)}, Ru(y)\rangle , \tag{13}$$

where T is some linear form on a space of functions on Y. Then, replacing u by the K-invariant function u_g in the lemma, we obtain

$$u(g \cdot x_o) = u_g(x_o) = \langle T, Ru_g\rangle .$$

The roles of g and t can now be interchanged by Lemma 11, whence

$$u(x) = \langle T_{(t)}, R_t^* Ru(x)\rangle , \tag{14}$$

for arbitrary $u \in \mathcal{D}(X)$ and $x \in X$. The notation $T_{(t)}$ means that T now acts on the shift variable t, or $t \cdot y_o$ to be precise. Since $R_{kth}^* Ru(x) = R_t^* Ru(x)$ for $k \in K$ and $h \in H$, this variable may actually be taken in $K\backslash G/H$.

The general inversion formula (14) *for* R *thus follows from the special case* (13) *of* K*-invariant functions at the origin*, thanks to the shifted dual transform.

If X is an isotropic space, the above trick (replace u by u_g) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

6.3 EXAMPLES

a. HOROCYCLE TRANSFORM. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, **d**), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in X, resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o)\, dn$$

(integrating over M is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o)\, dk\,.$$

For K-invariant u the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o)\, dn = a^{-\rho} \mathcal{A}u(a)\,;$$

the *Abel transform* $\mathcal{A}$ is defined by this equality.

For K-invariant $u \in \mathcal{D}(X)$ we have $\mathcal{A}u \in \mathcal{D}(A)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. It is known from spherical harmonic analysis on X that the classical Fourier transform

$$\widehat{\mathcal{A}u}(\lambda) = \int_A a^{-i\lambda} \mathcal{A}u(a)\, da\,, \quad \lambda \in \mathfrak{a}^*\,,$$

coincides with the spherical transform of u, with the inversion formula ([9] p. 454)

$$u(x_o) = C \int_{\mathfrak{a}^*} \widehat{\mathcal{A}u}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda\,, \tag{15}$$

where C is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since

$C \cdot |c(\lambda)|^{-2}$ has polynomial growth on $\mathfrak{a}^*$ its Fourier transform is a tempered distribution T on $A = \exp \mathfrak{a}$ such that

$$u(x_o) = \langle T, \mathcal{A}u \rangle = \langle T_{(a)}, a^\rho Ru(a \cdot y_o) \rangle .$$

Thus T inverts the Abel transform at the origin. By (14) we obtain the next theorem.

THEOREM 13. *Let X be a Riemannian symmetric space of the noncompact type. Its horocycle Radon transform R can be inverted by*

$$u(x) = \langle T_{(a)}, a^\rho R_a^* Ru(x) \rangle, \ x \in X ,$$

for $u \in \mathcal{D}(X)$. The distribution $T_{(a)}$ (acting on the variable $a \in A$) is, up to a constant factor, the Fourier transform of $|c(\lambda)|^{-2}$.

REMARKS.

(i) This extends a result by Berenstein and Tarabusi [2] for $X = H^n(\mathbf{R})$, obtained by direct calculations.

(ii) Helgason's original inversion formula ([11], p. 116)

$$u(x) = R^* \Lambda\overline{\Lambda} Ru(x)$$

follows easily from Theorem 13. Indeed Helgason's operator $\Lambda\overline{\Lambda}$ is defined as follows ([11], p. 111). Given $v \in \mathcal{D}(Y)$ and $g = kan \in G$, multiply $v(g \cdot y_o) = v(ka \cdot y_o)$ by a^ρ, take the Fourier transform with respect to $a \in A$, multiply it by $C \cdot |c(\lambda)|^{-2}$ (an even function of λ), take the inverse Fourier transform, and multiply by $a^{-\rho}$; the result is $\Lambda\overline{\Lambda} v(g \cdot y_o)$. In other words

$$\Lambda\overline{\Lambda} v(g \cdot y_o) = \Lambda\overline{\Lambda} v(ka \cdot y_o) = a^{-\rho} \left(T * (a^\rho v)\right)(ka \cdot y_o) ,$$

where $*$ is the convolution on A with respect to a. Let b denote a variable in A; since T is even we have

$$\begin{aligned} \Lambda\overline{\Lambda} v(g \cdot y_o) &= a^{-\rho} \langle T_{(b)}, (ab)^\rho v(kab \cdot y_o) \rangle \\ &= \langle T_{(b)}, b^\rho v(kab \cdot y_o) \rangle = \langle T_{(b)}, b^\rho v(gb \cdot y_o) \rangle . \end{aligned}$$

Replacing v by Ru, g by gk and integrating with respect to $k \in K$ we obtain

$$\begin{aligned} R^* \Lambda\overline{\Lambda} Ru(g \cdot x_o) &= \int_K \langle T_{(b)}, b^\rho Ru(gkb \cdot y_o) \rangle dk \\ &= \left\langle T_{(b)}, b^\rho \int_K Ru(gkb \cdot y_o)\, dk \right\rangle = \langle T_{(b)}, b^\rho R_b^* Ru(g \cdot x_o) \rangle . \end{aligned}$$

By Theorem 13 this is $u(g \cdot x_o)$, as claimed.

(iii) Note that T is supported at the origin if and only if $|c(\lambda)|^{-2}$ is a polynomial, i.e. if the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below).

b. TOTALLY GEODESIC TRANSFORM ON CLASSICAL HYPERBOLIC SPACES. We retain the notation of Section 4.3 **c**.

THEOREM 14. *Let $X = H^m(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, be one of the classical hyperbolic spaces, let $\mathfrak{s}$ be any $\mathbf{F}$-vector subspace of $\mathfrak{p} = \mathbf{F}^m$, and T any unit vector orthogonal to $\mathfrak{s}$ in $\mathfrak{p}$.*

For the Radon transform defined by the totally geodesic submanifolds $y = g\cdot \operatorname{Exp}\mathfrak{s}$, of (real) dimension d, we have the following inversion formulas by means of shifted dual transforms, for $u \in \mathcal{D}(X)$ and $x \in X$.

(i) *If $d = 2k+1$ is odd, $k \geq 0$,*

$$2^k\pi^{k+1}u(x) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1}\int_0^\sigma \left(R^*_{\exp t(\tau)T}Ru(x)\right)(\sigma^2-\tau^2)^{-1/2}d\tau\Big|_{\sigma=1},$$

where $t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$.

(ii) *If $d = 2k$ is even, $k \geq 1$, there exists a polynomial of degree k*

$$Q_k(\lambda) = \frac{2^k k!}{(2k)!}\lambda^k + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1),$$

with rational coefficients (depending on k and $q' = \dim\mathfrak{s}_{2\alpha}$), such that

$$(-2\pi)^k u(x) = Q_k(\partial_t^2)\left(R^*_{\exp tT}Ru(x)\right)_{t=0}.$$

REMARKS. This extends a result proved by Helgason ([10], p. 144, or [14], p. 97) for $\mathbf{F} = \mathbf{R}$. In case (i), a look at the proof below shows that an arbitrary positive integer ℓ may be added to the exponents of $\sigma^{-1}\partial_\sigma$ and $\sigma^2-\tau^2$; Helgason's result is obtained for $\ell = k$. From the proof of case (ii) we obtain for $k = 1, 2$

$$Q_1(\partial_t^2) = \partial_t^2 + q' + 1$$

$$Q_2(\partial_t^2) = \frac{1}{3}\partial_t^4 + \left(2q' + \frac{14}{3}\right)\partial_t^2 + (q'+1)(q'+3).$$

Our d is of course even whenever $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$. A comparison with Section 4.3 **c** shows that (except for $\mathbf{F} = \mathbf{R}$) the present assumption on $\mathfrak{s}$ is stronger than in Theorem 8.

Proof of Theorem 14. In order to use spherical coordinates on totally geodesic submanifolds of X, we need a lemma. As in Section 4.3 **c**, the matrices in $\mathfrak{p}$ can be identified to vectors $V = (V_1, \dots, V_m) \in \mathbf{F}^m$, and the scalar product of $T, V \in \mathfrak{p}$ is

$$(T, V) = \mathrm{Re}\left(\bar{T} \cdot V\right), \quad \text{with } \bar{T} \cdot V = \sum_{i=1}^{m} \bar{T}_i V_i .$$

Let $\| \ \|$ be the corresponding norm.

LEMMA 15. *Let $X = H^m(\mathbf{F})$ be a classical hyperbolic space.*

(i) *Let $T, V \in \mathfrak{p}$. In the geodesic triangle with vertices x_o (the origin of X), $\mathrm{Exp}\, T$ and $\exp T \cdot \mathrm{Exp}\, V$, the Riemannian lengths of the sides are $t = \|T\|$, $r = \|V\|$ and w given by*

$$\cosh^2 w = \left(\cosh t \cosh r + \frac{\sinh t}{t}\frac{\sinh r}{r}(T, V)\right)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\bar{T} \cdot V - (T, V)\right|\right)^2 .$$

(ii) *Let $\mathfrak{s} \subset \mathfrak{p}$ be a Lie triple system. If $T \in \mathfrak{p}$ is orthogonal to $\mathfrak{s}$, the totally geodesic submanifold $\exp T \cdot \mathrm{Exp}\, \mathfrak{s}$ is at distance $t = \|T\|$ from the origin.*

Proof. (i) The Riemannian distance from x_o to $\mathrm{Exp}\, T$ is $\|T\| = t$. Transforming x_o and $\mathrm{Exp}\, V$ by the isometry $\exp T \in G$ shows that the second side of the triangle has length r. The third side is $w = \|W\|$, where W is the unique $W \in \mathfrak{p}$ such that $\mathrm{Exp}\, W = \exp T \cdot \mathrm{Exp}\, V$, in other words

$$\exp W = (\exp T \exp V)\, k$$

for some $k \in K$. The map $g \mapsto g\theta(g)^{-1}$, where θ is the Cartan involution of G, transforms this equality into

$$\exp 2W = \exp T \exp 2V \exp T .$$

By elementary matrix computations $T^3 = t^2 T$, and the exponential is

$$\exp T = I + \frac{\sinh t}{t} T + \frac{\cosh t - 1}{t^2} T^2 ,$$

where I is the unit matrix. Now $\mathrm{tr}\, T = 0$ and $\mathrm{tr}\, T^2 = 2t^2$ is real, so that taking the traces we obtain

$$\mathrm{tr}\,(\exp 2W) = \mathrm{Re}\,\mathrm{tr}\,(\exp 2W) = \mathrm{Re}\,\mathrm{tr}\,(\exp 2T \exp 2V) ;$$

indeed $\mathrm{Re}\,\mathrm{tr}\,(gg') = \mathrm{Re}\,\mathrm{tr}\,(g'g)$ for $g, g' \in G$, even when $\mathbf{F} = \mathbf{H}$.

Taking account of

$$\operatorname{Re}\operatorname{tr} TV = 2(T,V), \qquad \operatorname{tr} T^2V = \operatorname{tr} TV^2 = 0,$$
$$\operatorname{Re}\operatorname{tr} T^2V^2 = t^2r^2 + \left|\overline{T}\cdot V\right|^2,$$

the expression of $\cosh w$ follows after some elementary calculations.

(ii) Let $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$. By (i) with $V \in \mathfrak{s}$ and $(T,V) = 0$, the distance w of the origin to the point $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$ of y is given by

$$\cosh^2 w = (\cosh t \cosh r)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\overline{T}\cdot V\right|\right)^2 .$$

Therefore $w \geq t$, with equality if and only if $V = 0$, and $\operatorname{Exp} T$ is the unique point of y closest to x_o (geodesic orthogonal projection of the origin on y). The lemma is proved. □

Going back to Theorem 14, let $g \in G$ and let $y = g \cdot \operatorname{Exp} \mathfrak{s}$ be an arbitrary given totally geodesic submanifold, element of Y. The minimum distance between y and the origin x_o is obtained at a point $\operatorname{Exp} T \in y$, with $T \in \mathfrak{p}$. In particular there exists $V \in \mathfrak{s}$ such that $\operatorname{Exp} T = g \cdot \operatorname{Exp} V$, i.e. $(\exp T)k = g \exp V$ for some $k \in K$. But $\operatorname{Exp} \mathfrak{s}$ is globally invariant under the action of $\exp V$, so that $y = (\exp T)k \cdot \operatorname{Exp} \mathfrak{s} = \exp T \cdot \operatorname{Exp}(k \cdot \mathfrak{s})$. Changing notation, we may write $\mathfrak{s}$ for $k \cdot \mathfrak{s}$ and $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$.

Let $V \in \mathfrak{s}$. On the geodesic $\exp T \cdot \operatorname{Exp} sV$, $s \in \mathbf{R}$, contained in y, the minimum distance to x_o is obtained for $s = 0$. By Lemma 15 (i) with sV instead of V, this implies $(T,V) = 0$ so that T is orthogonal to $\mathfrak{s}$ and Lemma 15 (ii) applies.

Besides, if we assume $\mathfrak{s}$ is a $\mathbf{F}$-vector subspace of $\mathfrak{p}$ therefore a Lie triple system (Section 4.3 **c**), the vector T must be orthogonal to all $V\lambda$, $V \in \mathfrak{s}$, $\lambda \in \mathbf{F}$, whence $\overline{T}\cdot V = 0$. By Lemma 15 the distance $w = w(t,r)$ between x_o and an arbitrary point $x = \exp T \cdot \operatorname{Exp} V$ of y is simply given by

$$\cosh w(t,r) = \cosh t \cosh r, \quad t = \|T\|, \quad r = \|V\|, \tag{16}$$

the same expression as for real hyperbolic spaces.

According to (13) and (14) we only need to invert R at the origin for a K-invariant function u. As shown in Section 4.1 **a**, Lemma 1 applies and $Ru(y) = \int_y u(x)\, dm_y$. When u is radial the integral can be obtained in spherical coordinates on y with origin $\operatorname{Exp} T$, as

$$Ru(y) = \int_0^\infty u(w(t,r))A_o(r)\, dr \tag{17}$$

where $A_o(r) = \omega_d(\sinh r)^{d-1}(\cosh r)^{q'}$ is the area of spheres of radius r in $\mathcal{Y}$. By (16) and (17) Ru may be viewed as a smooth even function $Ru(t)$ of $t \in \mathbf{R}$.

The end of the proof is now similar to the case of $H^n(\mathbf{R})$, as given in [11], p. 53 or [14], p. 97. Let $\tau = (\cosh t)^{-1}$, and let $t = t(\tau) \geq 0$ denote the inverse function. Introducing the functions

$$\varphi(\tau) = \tau^{-d-q'} u(t(\tau)), \quad \psi(\tau) = \tau^{-1-q'} Ru(t(\tau)),$$

which are C^∞ on $]0, 1]$, (17) becomes

$$\psi(\tau) = \omega_d \int_0^\tau \varphi(\rho) \left(\tau^2 - \rho^2\right)^{(d/2)-1} d\rho. \tag{18}$$

Proof of (i). The Abel type integral equation (18) can be inverted as usual: it implies that, for any $a > 0$, $\sigma > 0$,

$$\Gamma\left(\frac{d}{2} + a\right) \int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau =$$

$$= \pi^{d/2}\Gamma(a) \int_0^\sigma \varphi(\rho) \left(\sigma^2 - \rho^2\right)^{(d/2)+a-1} d\rho$$

and, choosing $a > 0$ such that $N = (d/2) + a$ is a strictly positive integer, it follows easily that

$$2^{N-1}\pi^{d/2}\Gamma(a)\varphi(\sigma) = \sigma \left(\sigma^{-1}\partial_\sigma\right)^N \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau\right).$$

If $d = 2k + 1$ is odd, $k \geq 0$, the smallest such a is $1/2$ so that $N = k + 1$ and

$$2^k\pi^{k+1}\varphi(\sigma) = \sigma \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{-1/2} \tau d\tau\right), \quad \sigma > 0;$$

the derivatives cannot be taken here under the integral. Besides d can only be odd for $\mathbf{F} = \mathbf{R}$ according to the assumption on $\mathfrak{s}$, and $q' = 0$ in that case. Going back to u and Ru we thus obtain for $\sigma = 1$

$$2^k\pi^{k+1}u(x_o) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \int_0^\sigma Ru(t(\tau))(\sigma^2 - \tau^2)^{-1/2} d\tau \Big|_{\sigma=1},$$

for any K-invariant $u \in \mathcal{D}(X)$. The claim follows by Section 6.2.

Proof of (ii). If $d = 2k$ is even, $k \geq 1$, the integral equation (18) can be directly solved as

$$(2\pi)^k\varphi(\tau) = \tau \left(\tau^{-1}\partial_\tau\right)^k \psi(\tau), \quad \tau > 0.$$

In particular, at the origin,

$$(2\pi)^k u(x_o) = \left(\tau^{-1}\partial_\tau\right)^k \left(\tau^{-1-q'} Ru(t(\tau))\right)_{\tau=1}$$
$$= \left(\partial_\tau^k + \cdots + (-1)^k (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t(\tau))\Big|_{\tau=1} .$$

To switch over to derivatives with respect to t we note that, if $g(\tau) = f(t)$ with $\tau = (\cosh t)^{-1} = 1 - \frac{t^2}{2} + \cdots$, identification of Taylor expansions at $\tau = 1$, resp. $t = 0$, leads to

$$\left(-\frac{1}{2}\right)^k \frac{g^{(k)}(1)}{k!} = \frac{f^{(2k)}(0)}{(2k)!} + \cdots + a_k f''(0) ,$$

where dots are a sum of even derivatives of f multiplied by some rational coefficients (like a_k). Therefore

$$(-2\pi)^k u(x_o) = \left(\frac{2^k k!}{(2k)!}\partial_t^{2k} + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t)|_{t=0} ,$$

for any K-invariant $u \in \mathcal{D}(X)$, whence the claim by Section 6.2. □

6.4 The amusing formula generalized

a. To motivate the forthcoming generalizations of the amusing formula (12) and their applications to Radon inversion, we briefly recall the classical example of points and hyperplanes in the Euclidean space $X = \mathbf{R}^n$. Let (ω, p) be parameters for the hyperplane defined by the equation $\omega \cdot x = p$, where ω is a unit vector, p is a real number and $\cdot$ is the scalar product. Given $t \in \mathbf{R}$ and a point $x \in \mathbf{R}^n$, the parameters $(\omega, p) = (\omega, t + \omega \cdot x)$ define a hyperplane at distance $|t|$ from x, and

$$R_t^* v(x) = \int_{S^{n-1}} v(\omega, t + \omega \cdot x)\, d\omega$$

is the corresponding shifted dual Radon transform, where $v(\omega, p) = v(-\omega, -p)$ is an arbitrary smooth even function on $S^{n-1} \times \mathbf{R}$. Changing ω into $-\omega$ in the integral shows that $R_t^* v(x)$ is an even function of t.

Since $\sum \omega_i^2 = 1$ it is easily checked that

$$\left(\partial_t^2 - \Delta_x\right) v(\omega, t + \omega \cdot x) = 0 ,$$

where Δ_x is the Euclidean Laplace operator acting on x. Thus $R_t^* v(x)$, as a function of (x, t) in $\mathbf{R}^n \times \mathbf{R}$, is a solution of the wave equation, being an integral of the *elementary plane waves* $v(\omega, t+\omega\cdot x)$. More generally, for any positive integer k,

$$(\partial_t^{2k} - \Delta_x^k)\, R_t^* v(x) = 0\,. \tag{19}$$

For odd n we have, by Theorem 8 with $n = 2k+1$, $d = 2k$ and $\varepsilon = 0$, the following inversion formula for the Radon transform on hyperplanes

$$Cu(x) = \Delta_x^k R^* Ru(x)\,. \tag{20}$$

Putting $v = Ru$ in (19) and observing that $R^* = R_0^*$, we thus obtain a new inversion formula by means of the shifted dual transform

$$Cu(x) = \partial_t^{n-1} R_t^* Ru(x)|_{t=0}\,. \tag{21}$$

Formula (21) might also be proved directly by the method of Section 6.2.

b. To extend formula (12) we first deal with the Laplace operator; general invariant operators will be considered in the next section.

Let G be a Lie group, K a compact subgroup and let L be the Laplace operator of the Riemannian manifold $X = G/K$ (cf. Notations, **b**). The operator L can be expressed by means of any orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{p}$ as

$$Lf(gK) = \sum_{j=1}^{n} \partial_s^2 f\left(g \exp(sX_j)K\right)\big|_{s=0}\,,$$

with $f \in C^2(G/K)$, $g \in G$; indeed both sides are G-invariant operators on X which coincide at $g = e$.

Now let $Y = G/H$ where H is a Lie subgroup of G and, as before,

$$R^* v(gK) = \int_K v(gkH)\, dk\,, \quad R_t^* v(gK) = \int_K v(gktH)\, dk$$

for $v \in C^2(Y)$ and $g, t \in G$. Then

$$LR^* v(gK) = \int_K \Big(\sum_j \partial_s^2 v\left(g \exp(sX_j)kH\right)\big|_{s=0}\Big)\, dk\,. \tag{22}$$

But $\sum X_j^2$ is a K-invariant element in the symmetric algebra of $\mathfrak{p}$ and it follows that, for any $\varphi \in C^2(\mathfrak{p})$, $k \in K$,

$$\sum_j \partial_s^2 \varphi\,(sX_j)\big|_{s=0} = \sum_j \partial_s^2 \varphi\left(s\,(k\cdot X_j)\right)\Big|_{s=0}\,.$$

Therefore k can be moved to the left of $\exp sX_j$ in (22) and we obtain

$$LR^*v(x) = \sum_j \partial_s^2 R^*_{\exp sX_j} v(x)\Big|_{s=0} \tag{23}$$

for $v \in C^2(Y)$, $x \in X$. If $\mathfrak{h} \cap \mathfrak{p}$ is a nontrivial subspace of $\mathfrak{p}$ and the basis (X_j) contains a basis of this subspace, the sum in (23) only runs over an orthonormal basis of the orthogonal subspace $(\mathfrak{h} \cap \mathfrak{p})^\perp$, due to the right H-invariance of v.

We now give a more specific result for the *geodesic Radon transform*, in the notation of Section 4.1. If $\mathfrak{s}$ is a d-dimensional Lie triple system contained in $\mathfrak{p}$ and $y_o = \operatorname{Exp} \mathfrak{s}$ the corresponding totally geodesic submanifold of X, we take as Y the set of all $g \cdot y_o$ for $g \in G$. Then $Y = G/H$, where H is the subgroup of all $h \in G$ globally preserving y_o.

PROPOSITION 16. *Let X be one of the classical hyperbolic spaces $H^n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Assume $\mathfrak{s}$ is a $\mathbf{F}$-vector subspace of $\mathfrak{p}$ and let $T \in \mathfrak{p}$ be any unit vector orthogonal to $\mathfrak{s}$. For $v \in C^2(Y)$, the shifted dual geodesic transform $R^*_{\exp tT} v$ is then an even function of $t \in \mathbf{R}$ and, for $x \in X$,*

$$LR^*v(x) = (n-d)\, \partial_t^2 R^*_{\exp tT} v(x)\Big|_{t=0}$$

where n and d denote the real *dimension of X and $\mathfrak{s}$ respectively.*

In other words, the function $(x,t) \mapsto R^*_{\exp tT} v(x)$ is a solution at time $t = 0$ of the wave operator $L - (n-d)\partial_t^2$ on $X \times \mathbf{R}$.

Applying the proposition to $H^3(\mathbf{R})$ with $d = 2$ we obtain formula (12). Indeed, if $\varphi(t)$ is an even function of t, let ψ be defined by $\psi(\tau) = \varphi(t)$ with $\cosh t = 1/\tau$; then $-\psi'(1) = \varphi''(0)$.

EXAMPLE. By Theorem 8 the 2-geodesic transform on $X = H^n(\mathbf{R})$ can be inverted by means of a second order differential operator:

$$-2\pi(n-2)u = (L + n - 2)R^*Ru\,,$$

and Proposition 16 now yields the inversion formula

$$-2\pi u = \left(\partial_t^2 + 1\right) R^*_{\exp tT} Ru\Big|_{t=0}\,, \tag{24}$$

where $u \in \mathcal{D}(X)$ and $T \in \mathfrak{p}$ is any unit vector orthogonal to $\mathfrak{s}$. Formula (24) also follows from Theorem 14 (ii) with $k = 1$, $q' = 0$.

Proof of Proposition 16. The point is to show that the group $K \cap H$ acts transitively on the unit sphere of $\mathfrak{s}^\perp$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$. For the scalar product $(T,V) = \mathrm{Re} \sum \bar{T}_i V_i$ on $\mathfrak{p}$ we have $(T, V\lambda) = (T\bar{\lambda}, V)$, $\lambda \in \mathbf{F}$, therefore $\mathfrak{s}^\perp$ is a $\mathbf{F}$-subspace of $\mathfrak{p}$.

An element k of $K \cap H$ is characterized by $k \in K$ and $k \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$, i.e. $k \cdot \mathfrak{s} = \mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of $\mathfrak{p}$ and $\mathfrak{s}$ as $\mathbf{F}$-vector spaces. Taking a $\mathbf{F}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, it follows that

$$K = U(1;\mathbf{F}) \times U(n';\mathbf{F}), \quad K \cap H = U(1;\mathbf{F}) \times U(d';\mathbf{F}) \times U(n'-d';\mathbf{F}).$$

But $U(n'-d';\mathbf{F})$ acts transitively on the unit sphere of $\mathbf{F}^{n'-d'}$, which implies our claim.

If $T, T' \in \mathfrak{s}^\perp$ are two unit vectors, there exists $k_o \in K \cap H$ such that $k_o \cdot T = T'$. Thus

$$\begin{aligned} R^*_{\exp tT'} v(gK) &= \int_K v\left(gkk_o \exp(tT) k_o^{-1} H\right) dk \\ &= \int_K v\left(gk \exp(tT) H\right) dk = R^*_{\exp tT} v(gK). \end{aligned}$$

In particular $R^*_{\exp tT} v$ is an even function of t.

Going back to (23), we now take as (X_j) an orthonormal $\mathbf{R}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. The $n-d$ basis vectors in $\mathfrak{s}^\perp$ give the same contribution to the right hand side, whereas the d vectors in $\mathfrak{s}$ generate one parameters subgroups of H and give no contribution; indeed $\exp tV \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$ for $V \in \mathfrak{s}$, since $\mathfrak{s}$ is a Lie triple system by Section 4.3 **c**. This completes the proof. □

6.5 Multitemporal waves

We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X = G/K$, $Y = G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the respective Lie algebras, and $\mathfrak{t}$ a vector subspace of $\mathfrak{g}$ such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let $K_1, \dots, K_p$ be a basis of $\mathfrak{k}$, complemented by $H_1, \dots, H_q \in \mathfrak{h}$ so that the K_i's and H_j's are a basis of $\mathfrak{k} + \mathfrak{h}$, and let $T_1, \dots, T_r$ be a basis of $\mathfrak{t}$. We shall use the same notations for the corresponding left-invariant vector fields on G, e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with $f \in C^\infty(G)$, $g \in G$, $s \in \mathbf{R}$. We denote by $\mathbf{D}(G)$ the algebra of all left invariant differential operators on G, by $\mathbf{D}(G)^K$ the subalgebra of right

K-invariant operators and by $\mathbf{D}(X)$ the algebra of G-invariant differential operators on X. For $s = (s_1, \dots, s_r) \in \mathbf{R}^r$, let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r\,.$$

We recall that, for $g, t \in G$,

$$R_t^* v(gK) = \int_K v(gktH)\, dk\,.$$

THEOREM 17. *Let G be a Lie group, H, K Lie subgroups, with K compact and $X = G/K$, $Y = G/H$.*

(i) *For any $P \in \mathbf{D}(X)$ there exists $Q(\partial)$, a constant coefficients differential operator on $\mathbf{R}^r$, with order(Q) $\leq$ order(P), such that for any $v \in C^\infty(Y)$, $x \in X$,*

$$PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x)\Big|_{s=0}\,. \tag{25}$$

(ii) *Assume furthermore that $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$, and let T denote the connected Lie subgroup of G with Lie algebra $\mathfrak{t}$. Then for any $P \in \mathbf{D}(X)$ there exists a right-invariant differential operator Q on T, with order(Q) $\leq$ order(P), such that*

$$P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x) \tag{26}$$

for $v \in C^\infty(Y)$; here $P_{(x)}$ acts on the variable $x \in X$ and $Q_{(t)}$ acts on $t \in T$.

Thus $R_t^* v(x)$, as a function of $(x, t) \in X \times T$, solves the generalized "*multitemporal*" wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables $(x, s) \in X \times \mathbf{R}^r$ at the time $s = 0$.

Proof. In order to work on G rather than on its homogeneous spaces, we define $w(g) = v(gH)$ and, for $g, t \in G$,

$$F(g, t) = \left(R_t^* v\right)(gK) = \int_K w(gkt)\, dk\,, \tag{27}$$

so that $F(gk, k'th) = F(g, t)$ for any $k, k' \in K$, $h \in H$, and

$$F(g, e) = \left(R^* v\right)(gK) = \int_K w(gk)\, dk\,.$$

Let $P \in \mathbf{D}(X)$ be given. Since K is compact the coset space $X = G/K$ is reductive and there exists $D \in \mathbf{D}(G)^K$ such that ([9], p. 285)

$$(Pf)(gK) = D_{(g)}(f(gK)) \tag{28}$$

for $f \in C^\infty(X)$, $g \in G$.

To transfer derivatives from g to t we observe that, by the invariance of D under left translation by gk and right translation by k,

$$D_{(g)} w(gkt) = D_{(x)} w(gkxt)\big|_{x=e} ,$$

where g, x, t are variables in G. Integrating over K it follows that

$$D_{(g)} F(g,t) = D_{(x)} F(g,xt)\big|_{x=e} , \tag{29}$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \cdots K_p^{\beta_p} T_1^{\alpha_1} \cdots T_r^{\alpha_r} H_1^{\gamma_1} \cdots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of $\mathbf{D}(G)$. Setting apart the terms with $\beta = \gamma = 0$, we can thus write, for some $E_i, F_j \in \mathbf{D}(G)$ and some constant coefficients a_α,

$$D = D' + \sum_{i=1}^{p} K_i E_i + \sum_{j=1}^{q} F_j H_j , \quad D' = \sum_\alpha a_{\alpha_1 \ldots \alpha_r} T_1^{\alpha_1} \cdots T_r^{\alpha_r} . \tag{30}$$

If we replace $D_{(x)}$ by (30) in (29), the second term $(K_i E_i)_{(x)} F(g,xt)\big|_{x=e}$ vanishes because $K_i \in \mathfrak{k}$ and $F(g,kxt) = F(g,t)$. In the third term the left invariant vector field $H_j \in\in \mathfrak{h}$ acts by

$$(H_j)_{(x)} F(g,xt) = \partial_s F(g, x \exp\left(sH_j\right) t)\big|_{s=0} ,$$

and this vanishes too whenever t normalizes H, because $F(g,xth) = F(g,xt)$.

Since $t = e$ in case (i), or $t \in T$ with $Ht = tH$ in case (ii), we finally obtain for both cases (in multi-index notation)

$$\begin{aligned} D_{(g)} F(g,t) &= D'_{(x)} F(g,xt)\big|_{x=e} \\ &= \sum_\alpha a_\alpha \partial_s^\alpha F(g, (\exp s_1 T_1 \cdots \exp s_r T_r)\, t)\Big|_{s=0} \\ &= \left(\sum_\alpha a_\alpha \partial_s^\alpha \right) F(g, t(s)t)\Big|_{s=0} . \end{aligned} \tag{31}$$

Let the operator Q be defined by

$$Qf(t) = \sum_\alpha a_\alpha \partial_s^\alpha f(t(s)t)\Big|_{s=0} ,$$

a right invariant differential operator on the group T in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii). □

6.6 EXAMPLES

Keeping the notations of the previous section, we shall illustrate Theorem 17.

a. TOTALLY GEODESIC TRANSFORM. As in Section 4.1 **a**, let $X = G/K$ be a Riemannian symmetric space of the noncompact type and $y_o = \operatorname{Exp} \mathfrak{s}$ the origin in the dual space $Y = G/H$. By (3) we have $\mathfrak{k}+\mathfrak{h} = \mathfrak{k}\oplus\mathfrak{s}$, therefore Theorem 17 (i) applies with $\mathfrak{t} = \mathfrak{s}^{\perp}$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$.

b. HOROCYCLE TRANSFORM. Again $X = G/K$ is a Riemannian symmetric space of the noncompact type (see Notations, **d**), but the dual space is now the space of horocycles $Y = G/MN$. We recall Harish-Chandra's isomorphism of algebras ([9], p. 306)

$$\Gamma : \mathbf{D}(X) \longrightarrow \mathbf{D}(A)^W ,$$

where $\mathbf{D}(A)^W$ is the subalgebra of W-invariant differential operators in $\mathbf{D}(A)$. The definition of Γ will be recalled during the next proof.

PROPOSITION 18. *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$w(x,a) = a^\rho R_a^* v(x) = a^\rho \int_K v(gkaN)\, dk$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x,a) = \Gamma(P)_{(a)} w(x,a) , \quad P \in \mathbf{D}(X), x \in X, a \in A .$$

Proof. Theorem 17 (ii) applies here with $T = A$, the abelian subgroup from the Iwasawa decomposition $G = KAN$; indeed $\mathfrak{k}+\mathfrak{h} = \mathfrak{k}+\mathfrak{m}+\mathfrak{n} = \mathfrak{k}\oplus\mathfrak{n}$, and $\mathfrak{g} = (\mathfrak{k} \oplus \mathfrak{n}) \oplus \mathfrak{a}$, $[\mathfrak{a},\mathfrak{h}] \subset [\mathfrak{a},\mathfrak{m}]+ [\mathfrak{a},\mathfrak{n}] \subset \mathfrak{n} \subset \mathfrak{h}$. By (31) we thus have

$$P_{(x)} R_a^* v(x) = D'_{(a)} R_a^* v(x) , \tag{32}$$

where $D \in \mathbf{D}(G)^K$ is related to P by (28) and $D' \in \mathbf{D}(A)$ was characterized by

$$D - D' \in \mathfrak{k}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{n} . \tag{33}$$

To compare D' and $\Gamma(P)$ we recall that $\Gamma(P) = a^{-\rho} D_{\mathfrak{a}} \circ a^\rho$, where $D_{\mathfrak{a}} \in \mathbf{D}(A)$ is characterized by

$$D - D_{\mathfrak{a}} \in \mathfrak{n}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{k} . \tag{34}$$

Moreover $(Df)(a) = D_{\mathfrak{a}}(f(a))$ for $a \in A$, if $f \in C^\infty(G)$ is such that $f(ngk) = f(g)$ for any $g \in G$, $k \in K$, $n \in N$ ([9], p. 302 sq.).

Taking $u \in \mathcal{D}(G)$ we have, by a classical integral formula,

$$(35) \qquad \int_G Df(g) \cdot u(g)\, dg = \int_{N \times A \times K} Df(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk$$
$$= \int_{N \times A \times K} D_{\mathfrak{a}} f(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk\,.$$

On the other hand, this integral can be written with the transpose operator ${}^tD$ as

$$\int_G Df(g) \cdot u(g)\, dg = \int_G f(g)\, {}^tDu(g)\, dg$$
$$= \int_A f(a)\, a^{-2\rho} da \int_{N \times K} \left({}^tDu\right)(nak)\, dn\, dk\,.$$

But ${}^tD \in \mathbf{D}(G)^K$ therefore, for any $g \in G$,

$$\int_{N \times K} \left({}^tDu\right)(ngk)\, dn\, dk = \left({}^tD\right)_{(g)} \left(\int_{N \times K} u(ngk)\, dn\, dk \right)\,.$$

The latter integral, as a function of g, is left N-invariant and right K-invariant so that

$$\int_{N \times K} \left({}^tDu\right)(nak)\, dn\, dk = \left({}^tD\right)_{\mathfrak{a}} \left(\int_{N \times K} u(nak)\, dn\, dk \right)\,.$$

Since $\left({}^tD\right)_{\mathfrak{a}} = {}^t\left(D'\right)$ obviously by (33) and (34), we obtain

$$\int_G Df(g) \cdot u(g)\, dg = \int_A D'(f(a)a^{-2\rho})\, da \int_{N \times K} u(nak)\, dn\, dk$$
$$= \int_{N \times A \times K} \left(a^{2\rho} D' \circ a^{-2\rho}\right) f(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk\,,$$

for any $f \in C^\infty(A)$ and any $u \in \mathcal{D}(G)$. Comparing with (35) it follows that

$$D_{\mathfrak{a}} = a^{2\rho} D' \circ a^{-2\rho}\,, \quad D' = a^{-\rho} \Gamma(P) \circ a^{\rho}\,,$$

whence the result by (32). □

A slightly different proof can be obtained by decomposing the wave $a^\rho R_a^* v(gK)$ into *elementary horocycle waves* as follows. For $g \in G$ we denote by $A(g) \in A$ the A-component of g in the Iwasawa decompositions $G = NAK = ANK$ (we recall that A normalizes N), and by $K(g) \in K$ its K-component in the decompositions $G = KAN = KNA$.

PROPOSITION 19. (i) *Given $f \in C^\infty(A)$ and $k \in K$, the function*

$$w(gK, a) = a^{-\rho} f(A(k^{-1}g)a)$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x, a) = \Gamma(P)_{(a)} w(x, a), \quad P \in \mathbf{D}(X),\ x \in X,\ a \in A.$$

(ii) *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$a^\rho R_a^* v(gK) = \int_K a^\rho v(gkaN)\, dk$$

is a solution of the same equations.

REMARKS. Part (i) is Proposition 8.5 in [12], p. 118. Note that, k being fixed, the "wave surfaces" $A(k^{-1}g) = constant$ are parallel horocycles with the same normal $kM \in K/M$ (cf. [11], p. 81). Indeed the equality $A(k^{-1}g) = a_o \in A$ is equivalent to $k^{-1}g \in a_o NK$, i.e. $g \cdot x_o \in ka_o \cdot y_o$.

If λ is a linear form on $\mathfrak{a}$ and $f(a) = a^{i\lambda+\rho}$, the result (i) implies that $A(k^{-1}g)^{i\lambda+\rho}$ is, as a function of gK, an eigenfunction of all invariant operators $P \in \mathbf{D}(X)$; this is a fundamental result for harmonic analysis on X.

Part (ii) provides a simpler proof and a generalization of Proposition 8.6 in [12], p. 118, where v was the Radon transform Ru of some $u \in \mathcal{D}(X)$. We refer to [12] or [13] for a detailed study of those multitemporal wave equations.

Proof of Proposition 19. (i) Both sides of the wave equation are invariant under the action of K on X; we can therefore assume $k = e$. Now $w(gK, a) = a^{-\rho} f(A(g)a)$ is left N-invariant and right K-invariant as a function of g, and it will suffice to prove the result for $g = a \in A$.

By the decomposition (34) of D we have, for any $b \in A$,

$$D_{(g)} \left(f(A(g)b)\right)\Big|_{g=a} = (D_{\mathfrak{a}})_{(a)} (f(ab)) = a^\rho \Gamma(P)_{(a)} \left(a^{-\rho} f(ab)\right).$$

But $\Gamma(P)$ is an invariant differential operator on A, isomorphic to the additive group of a vector space, and we obtain

$$\begin{aligned} D_{(g)} \left(b^{-\rho} f(A(g)b)\right)\Big|_{g=a} &= a^\rho \Gamma(P)_{(a)} \left((ab)^{-\rho} f(ab)\right) \\ &= a^\rho \Gamma(P)_{(b)} \left((ab)^{-\rho} f(ab)\right) \\ &= \Gamma(P)_{(b)} \left(b^{-\rho} f(ab)\right) = \Gamma(P)_{(b)} \left(b^{-\rho} f(A(g)b)\right)\Big|_{g=a}. \end{aligned}$$

Thus (i) is proved for $g = a$.

(ii) Let $g \in G$, $k \in K$ and $k' = K(gk)$. Then $gk = k'a'n'$ with $a' \in A$ and $n' \in N$. It follows that $k'^{-1}g = a'n'k^{-1}$, therefore $a' = A(k'^{-1}g)$ and

$$gkaN = k'A\left(k'^{-1}g\right)aN\,.$$

For fixed g the map $k \mapsto K(gk) = k'$ is a diffeomorphism of K onto itself and, by the integral formula ([9], p. 197)

$$\int_K F(k')\,dk = \int_K A(k'^{-1}g)^{2\rho}\,F(k')\,dk'\,,$$

we have

$$\begin{aligned} a^\rho R_a^* v(gK) &= a^\rho \int_K v(gkaN)\,dk \\ &= a^\rho \int_K v(k'A(k'^{-1}g)aN)\,dk \\ &= a^{-\rho} \int_K \left(A(k'^{-1}g)a\right)^{2\rho}\, v(k'A(k'^{-1}g)aN)\,dk'\,. \end{aligned}$$

By (i) applied to the functions $f(a) = a^{2\rho}v(k'aN)$, $k' \in K$, this is a solution of the wave equations. □

COROLLARY 20 (Helgason). *If* $\mathfrak{g}$ *has only one conjugacy class of Cartan subalgebras, there exists a differential operator* $P \in \mathbf{D}(X)$ *such that the horocycle Radon transform of* $X = G/K$ *is inverted by*

$$u(x) = PR^*Ru(x)$$

for $u \in \mathcal{D}(X)$, $x \in X$.

We prove it here by means of shifted transforms and wave equations; see [11], p. 116 for Helgason's original proof.

Proof. The assumption on $\mathfrak{g}$ implies that, in the notation of (15), $C \cdot |c(\lambda)|^{-2}$ is a W-invariant polynomial on $\mathfrak{a}^*$. Let $P \in \mathbf{D}(X)$ be the corresponding operator under the isomorphism $\Gamma\colon \mathbf{D}(X) \to \mathbf{D}(A)^W$, so that $\Gamma(P)(i\lambda) = C \cdot |c(\lambda)|^{-2}$. By Theorem 13 and Proposition 19 (ii) (with $v = Ru$)

we have

$$u(x) = \langle T_{(a)}, a^\rho R_a^* Ru(x)\rangle = \Gamma(D)_{(a)}\left(a^\rho R_a^* Ru(x)\right)\big|_{a=e}$$
$$= P_{(x)}\left(a^\rho R_a^* Ru(x)\right)\big|_{a=e} = P_{(x)} R^* Ru(x)\,. \qquad \square$$

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