The compact cases (projective spaces) might be dealt with similarly. One should then normalize $H$ by $\alpha(H) = i$ and replace $\varepsilon$ by $i$. Integrals with respect to $r$ should run from 0 to the diameter $\ell$ of $X$, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

## 4.2  AN INVERSION FORMULA

The $G$-invariant differential operators on an isotropic space $X$ are the polynomials of its Laplace-Beltrami operator $L$ ([9], p. 288). In order to invert the $d$-geodesic Radon transform on $X$, Section 3.2 suggests looking for a polynomial $P$ such that the above distribution $S$ is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on $X$ defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a} (\cosh \varepsilon r)^{b} = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^{b},$$

where $a$ and $b$ are real constants and $r$ is the distance from the origin $x_o$; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q}, \quad S(r) = \left(\omega_d / \omega_n\right) f_{d-n, q'-q}$$

with $q$, $q'$, $n$ and $d$ as defined above.

PROPOSITION 6.  *Assume* $\varepsilon = 0$ *(Euclidean case), or* $\varepsilon = 1$ *and* $b = 0$, *or else* $\varepsilon = 1$ *and* $b = 1 - q$ *(hyperbolic cases). Then, for any integer* $k \geq 1$, *the function* $f_{2k-n,b}$ *defines a* $K$-*invariant distribution* $F_{2k-n,b}$ *on* $X$ *such that*

$$P_k(L) F_{2k-n,b} = \omega_n 2^{k-1} (k-1)! \, (2-n)(4-n) \cdots (2k-n) \overset{\bullet}{\delta},$$

*where* $\delta$ *is the Dirac distribution at the origin* $x_o$ *and* $P_k$ *is the polynomial*

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2 (n - 2j - b)(2j + b + q - 1)\right).$$

REMARK.  The case $b = 0$, $n = 2k + 2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.

*Proof.* By [9], p. 313 the radial part of $L$ is

$$\Delta = \partial_r^2 + \frac{A'(r)}{A(r)}\partial_r = A(r)^{-1} \circ \partial_r \circ A(r) \circ \partial_r$$

$$= \partial_r^2 + ((n-1)\varepsilon \coth \varepsilon r + q\varepsilon \tanh \varepsilon r)\, \partial_r$$

$$= \partial_r^2 + (p\varepsilon \coth \varepsilon r + 2q\varepsilon \coth 2\varepsilon r)\, \partial_r \,.$$

The proof of the proposition breaks up into a few easy facts. First we have, for any $a, b \in \mathbf{R}$, the following equality of functions of $r > 0$:

$$(6) \quad \left(\Delta - \varepsilon^2(a+b)(a+n+b+q-1)\right)f_{a,b}$$
$$= a(a+n-2)f_{a-2,b} - \varepsilon^2 b(b+q-1)f_{a,b-2}\,,$$

which is immediate from $\Delta f = A^{-1}(Af')'$ and the identities

$$f_{a,b}' = af_{a-1,b+1} + \varepsilon^2 bf_{a+1,b-1}\,, \quad f_{a,b} = f_{a,b-2} + \varepsilon^2 f_{a+2,b-2}\,.$$

LEMMA 7. *For $a + n \geq 2$, $\varepsilon = 0$ or $1$, the locally integrable function $f_{a,b}$ defines a $K$-invariant distribution $F_{a,b}$ on $X$ such that*

$$\left(L - \varepsilon^2(a+b)(a+n+b+q-1)\right)F_{a,b}$$
$$= \begin{cases} a(a+n-2)F_{a-2,b} - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n>2 \\ \omega_n a\, \delta - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n=2 \end{cases}$$

*(equality of distributions on $X$).*

EXAMPLE. Taking $b = 0$, resp. $b = 1 - q$, the lemma provides the following fundamental solutions (which coincide for $q = 1$)

$$\left(L + \varepsilon^2(n-2)(q+1)\right)\left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n} = \omega_n(2-n)\delta$$

$$\left(L + 2\varepsilon^2(n+q-3)\right)\left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n}(\cosh \varepsilon r)^{1-q} = \omega_n(2-n)\delta\,.$$

In the flat case $\varepsilon = 0$ they both reduce to $Lr^{2-n} = \omega_n(2-n)\delta$, a classical result for $\mathbf{R}^n$.

*Proof of Lemma 7.* Due to the $K$-invariance of $f_{a,b}$ and $L$ we need only consider $K$-invariant test functions $u \in \mathcal{D}(X)$. The integral

$$\int_X f_{a,b} \cdot u\, dx = \int_0^\infty f_{a,b}(r)u(r)A(r)\, dr = \omega_n \int_0^\infty f_{a+n-1,b+q}(r)u(r)\, dr\,,$$

absolutely convergent if $a + n > 0$, defines a distribution $F_{a,b}$ on $X$. In view of the symmetry and $K$-invariance of the Laplace operator we have

$$
\begin{aligned}
\langle LF_{a,b}, u \rangle &= \langle F_{a,b}, Lu \rangle \\
&= \int_0^\infty f_{a,b}(r)\Delta u(r)A(r)\, dr = \int_0^\infty f_{a,b}(Au')'\, dr \\
&= \left(Af'_{a,b}u\right)(0) - \left(Af_{a,b}u'\right)(0) + \int_0^\infty (Af'_{a,b})'u\, dr.
\end{aligned}
$$

If $a + n > 2$ the function $Af_{a,b}$ vanishes to order $a + n - 1$ at the origin, and $Af'_{a,b}$ to order $a + n - 2$. Since $u(r)$ is smooth (this notation stands for $u(\mathrm{Exp}\, rH)$ with $\|H\| = 1$), it follows that

$$
\langle LF_{a,b}, u \rangle = \int_0^\infty \Delta f_{a,b}(r)u(r)A(r)\, dr,
$$

whence the result by (6).

The case $a + n = 2$ is similar, in view of $(Af'_{a,b})(0) = \omega_n a$.   $\square$

Proposition 6 now follows easily: letting

$$
L_a = L - \varepsilon^2(a + b)(a + n + b + q - 1)
$$

we have, by Lemma 7,

$$
L_a F_{a,b} = \begin{cases} a(a + n - 2)F_{a-2,b} & \text{if } a + n > 2 \\ \omega_n a\, \delta & \text{if } a + n = 2 \end{cases}
$$

whenever $\varepsilon^2 b(b + q - 1) = 0$. Since

$$
P_k(L) = L_{2-n}L_{4-n} \cdots L_{2k-n},
$$

the proposition follows by induction on $k$.   $\square$

THEOREM 8.  *The $d$-geodesic Radon transform on a $n$-dimensional non-compact Riemannian isotropic space $X$ can be inverted by means of a polynomial of its Laplace-Beltrami operator $L$, under the following assumptions:*

(i)  *$d$ is even: $d = 2k$ with $k \geq 1$;*

(ii)  *$X = \mathbf{R}^n$, or $\dim \mathfrak{s}_{2\alpha} = \dim \mathfrak{p}_{2\alpha}$, or else $\dim \mathfrak{s}_{2\alpha} = 1$.*

*Then*

$$
Cu = P_k(L)R^*Ru,
$$

*for any $u \in \mathcal{D}(X)$, where $P_k$ is the polynomial from Proposition 6 (with $\varepsilon = 1$, $q = \dim \mathfrak{p}_{2\alpha}$ and $b + q = \dim \mathfrak{s}_{2\alpha}$ if $X$ is hyperbolic, or $\varepsilon = 0$ if $X = \mathbf{R}^n$) and*

$$
C = \omega_d(-1)^k 2^{k-1}(k - 1)!(n - 2)(n - 4) \cdots (n - 2k).
$$

*Proof.* By (5) one has $S = (\omega_d/\omega_n)f_{a,b}$, with $a = d - n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (Section 4.1 **b**). The theorem follows from Proposition 6 and Section 3.2. $\square$

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9], Chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial $P_k$ is related to representation theory.

## 4.3 EXAMPLES

According to assumption (ii), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbf{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space $Y$ consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbf{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

**a.** A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8], p. 335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is 2, 4 or 8; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

**b.** Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace $H$ by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbf{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}, \quad \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}.$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7], p. 171, or [9], p. 168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

**c.** TOTALLY GEODESIC TRANSFORM ON CLASSICAL HYPERBOLIC SPACES. Let $X = H^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1, m; \mathbf{F})$, $K = U(1; \mathbf{F}) \times U(m; \mathbf{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices