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$$u = (R^*Ru) * T.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G -invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ ,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \langle u(g \cdot x_o), \langle S, ({}^t D\varphi) \circ \tau(g) \rangle \rangle \quad \text{by (1)} \\ &= \langle u(g \cdot x_o), \langle S, {}^t D(\varphi \circ \tau(g)) \rangle \rangle, \end{aligned}$$

since the transpose operator ${}^t D$ is G -invariant too, as follows from the existence of a G -invariant measure on X . Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u(g \cdot x_o), \langle DS, \varphi \circ \tau(g) \rangle \rangle \\ &= \langle u * DS, \varphi \rangle, \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can find a G -invariant differential operator D on X with given fundamental solution S . We shall now investigate this question on the basis of Propositions 4 and 5.

4. RADON TRANSFORMS ON ISOTROPIC SPACES

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}), H^{2m}(\mathbf{C}), H^{4m}(\mathbf{H}), H^{16}(\mathbf{O}),$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the d -geodesic Radon transform on X , defined by integrating over a family of d -dimensional totally geodesic submanifolds of X . At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 TOTALLY GEODESIC SUBMANIFOLDS

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.

a. Let $X = G/K$ be a *Riemannian symmetric space of the noncompact type (of arbitrary rank)*, where G is a connected semisimple Lie group and K a maximal compact subgroup (see Notations, **c** and **d**).

At the Lie algebra level, a totally geodesic submanifold of X is defined by a *Lie triple system*, i.e. a vector subspace \mathfrak{s} of \mathfrak{p} such that $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Then $\text{Exp } \mathfrak{s}$ is totally geodesic in X and contains the origin x_o . Besides $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$ are Lie subalgebras of \mathfrak{g} . Let G' be the (closed) Lie subgroup with Lie algebra \mathfrak{g}' , and K' (with Lie algebra \mathfrak{k}') be the isotropy subgroup of x_o in G' . Then

$$\text{Exp } \mathfrak{s} = G'/K' = G' \cdot x_o,$$

a closed symmetric subspace of X ([8], p.224–226, or [15], p.234 sq.).

Now let Y be the set of all d -dimensional totally geodesic submanifolds $y = g \cdot y_o$ of X , with $g \in G$ and $y_o = \text{Exp } \mathfrak{s} = G' \cdot x_o$. *Lemma 1* applies: if H is the subgroup of all $h \in G$ such that $h \cdot y_o = y_o$, then $y_o = H \cdot x_o$, $Y = G/H$ and the incidence relation is $x \in y$.

It will be useful to note that the Lie algebra \mathfrak{h} of H satisfies

$$(3) \quad \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}), \quad \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s}, \mathfrak{s}], \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s}.$$

Indeed the definition of H shows its invariance under the Cartan involution of G , whence the direct sum decomposition of \mathfrak{h} . Besides \mathfrak{h} contains $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ by Lemma 1 and, for $V \in \mathfrak{h} \cap \mathfrak{p}$, the point $\exp V \cdot x_o = \text{Exp } V$ belongs to $H \cdot x_o = \text{Exp } \mathfrak{s}$, thus $V \in \mathfrak{s}$ by the injectivity of Exp on \mathfrak{p} .

By Lemma 1 the Radon transform of $u \in C_c(X)$ is given by

$$Ru(y) = \int_y u(x) dm_y(x) = \int_{\text{Exp } \mathfrak{s}} u(g \cdot x) dm_{y_o}(x),$$

where dm_{y_o} is the Riemannian measure induced by X on its submanifold $y_o = \text{Exp } \mathfrak{s}$.

b. RANK ONE CASE. We now restrict to the rank one case (hyperbolic spaces). Let $H \in \mathfrak{s}$ be a fixed non zero vector. The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of \mathfrak{p} and \mathfrak{s} , and $\text{Exp } \mathfrak{s}$ is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of $(\text{ad } H)^2$, with respective eigenvalues 0 , $(\alpha(H))^2$, $(2\alpha(H))^2$ (where α and 2α are the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$), implies a similar decomposition of the invariant subspace \mathfrak{s} :

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha},$$

with $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$ and $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$. We set

$$\begin{aligned} p &= \dim \mathfrak{p}_\alpha, & q &= \dim \mathfrak{p}_{2\alpha}, & n &= \dim X = p + q + 1, \\ p' &= \dim \mathfrak{s}_\alpha, & q' &= \dim \mathfrak{s}_{2\alpha}, & d &= \dim \mathfrak{s} = p' + q' + 1, \end{aligned}$$

with $q = q' = 0$ when 2α is not a root (case of real hyperbolic spaces).

Let us normalize the vector H by the condition $\alpha(H) = 1$. Multiplying if necessary the Riemannian metric of X by a constant factor, we may assume that the corresponding Euclidean norm on \mathfrak{p} satisfies $\|H\| = 1$. Since Exp is a diffeomorphism of \mathfrak{p} onto X , the integral of a function $u \in C_c(X)$ can be computed as

$$\int_X u(x) dx = \int_{\mathfrak{p}} u(\text{Exp } Z) J(Z) dZ,$$

where $J(Z) = \det_{\mathfrak{p}}(\sinh \text{ad } Z / \text{ad } Z)$ is the Jacobian of Exp , a K -invariant function on \mathfrak{p} . If u is K -invariant on X , we simply write $u(r)$ for $u(\text{Exp } Z) = u(\text{Exp } rH)$ with $r = \|Z\|$ whence, computing with spherical coordinates on \mathfrak{p} ,

$$\int_X u(x) dx = \int_0^\infty u(r) A(r) dr,$$

where $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \text{ad } rH / \text{ad } rH)$ is the area of the sphere with center x_o and radius r in X , and $\omega_n = 2\pi^{n/2} / \Gamma(n/2)$ is the area of the unit sphere in \mathbf{R}^n . Taking account of the eigenvalues of $(\text{ad } H)^2$ we obtain, with a parameter ε explained in the next remark,

$$(4) \quad A(r) = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^p \left(\frac{\sinh 2\varepsilon r}{2\varepsilon} \right)^q = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^{n-1} (\cosh \varepsilon r)^q.$$

A similar expression gives $A_o(r)$ for the submanifold y_o (with d, p', q' instead of n, p, q). The distribution S in Proposition 4 is thus defined by the radial function

$$(5) \quad S(r) = A_o(r)/A(r) = (\omega_d/\omega_n) \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^{d-n} (\cosh \varepsilon r)^{q'-q}.$$

REMARK. Here $\varepsilon = 1$ for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting $\varepsilon = 0$ and $(\sinh \varepsilon r)/\varepsilon = r$: when $X = \mathbf{R}^n$ the geodesic submanifolds are the affine d -planes, $1 \leq d \leq n-1$, and

$$S(r) = (\omega_d/\omega_n) r^{d-n}.$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize H by $\alpha(H) = i$ and replace ε by i . Integrals with respect to r should run from 0 to the diameter ℓ of X , i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

4.2 AN INVERSION FORMULA

The G -invariant differential operators on an isotropic space X are the polynomials of its Laplace-Beltrami operator L ([9], p. 288). In order to invert the d -geodesic Radon transform on X , Section 3.2 suggests looking for a polynomial P such that the above distribution S is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on X defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon} \right)^b,$$

where a and b are real constants and r is the distance from the origin x_o ; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q}, \quad S(r) = (\omega_d/\omega_n) f_{d-n,q'-q}$$

with q, q', n and d as defined above.

PROPOSITION 6. Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a K -invariant distribution $F_{2k-n,b}$ on X such that

$$P_k(L)F_{2k-n,b} = \omega_n 2^{k-1} (k-1)! (2-n)(4-n) \cdots (2k-n) \delta,$$

where δ is the Dirac distribution at the origin x_o and P_k is the polynomial

$$P_k(x) = \prod_{j=1}^k (x + \varepsilon^2(n - 2j - b)(2j + b + q - 1)).$$

REMARK. The case $b = 0, n = 2k + 2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.