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$$u = (R^* Ru) * T .$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G-invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ,

$$\begin{aligned} \langle D(u*S), \varphi\rangle &= \langle u * S, {}^t D\varphi\rangle \\ &= \big\langle u(g\cdot x_o), \langle S, ({}^tD\varphi)\circ\tau(g)\rangle\big\rangle \qquad \text{by (1)} \\ &= \big\langle u(g\cdot x_o), \langle S, {}^tD(\varphi\circ\tau(g))\rangle\big\rangle , \end{aligned}$$

since the transpose operator ${}^tD$ is G-invariant too, as follows from the existence of a G-invariant measure on X. Finally,

$$\begin{aligned} \langle D(u*S), \varphi\rangle &= \big\langle u(g\cdot x_o), \langle DS, \varphi\circ\tau(g)\rangle\big\rangle \\ &= \langle u * DS, \varphi\rangle , \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can *find a G-invariant differential operator D on X with given fundamental solution S*. We shall now investigate this question on the basis of Propositions 4 and 5.

4. Radon transforms on isotropic spaces

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}),\ H^{2m}(\mathbf{C}),\ H^{4m}(\mathbf{H}),\ H^{16}(\mathbf{O}) ,$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the d-geodesic Radon transform on X, defined by integrating over a family of d-dimensional totally geodesic submanifolds of X. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 Totally geodesic submanifolds

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.

a. Let $X = G/K$ be a *Riemannian symmetric space of the noncompact type (of arbitrary rank)*, where G is a connected semisimple Lie group and K a maximal compact subgroup (see Notations, **c** and **d**).

At the Lie algebra level, a totally geodesic submanifold of X is defined by a *Lie triple system*, i.e. a vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ such that $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Then $\mathrm{Exp}\, \mathfrak{s}$ is totally geodesic in X and contains the origin x_o. Besides $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$ are Lie subalgebras of $\mathfrak{g}$. Let G' be the (closed) Lie subgroup with Lie algebra $\mathfrak{g}'$, and K' (with Lie algebra $\mathfrak{k}'$) be the isotropy subgroup of x_o in G'. Then

$$\mathrm{Exp}\, \mathfrak{s} = G'/K' = G' \cdot x_o ,$$

a closed symmetric subspace of X ([8], p. 224–226, or [15], p. 234 sq.).

Now let Y be the set of all d-dimensional totally geodesic submanifolds $y = g \cdot y_o$ of X, with $g \in G$ and $y_o = \mathrm{Exp}\, \mathfrak{s} = G' \cdot x_o$. *Lemma 1 applies*: if H is the subgroup of all $h \in G$ such that $h \cdot y_o = y_o$, then $y_o = H \cdot x_o$, $Y = G/H$ and the incidence relation is $x \in y$.

It will be useful to note that the Lie algebra $\mathfrak{h}$ of H satisfies

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) , \quad \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s}, \mathfrak{s}] , \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s} . \tag{3}$$

Indeed the definition of H shows its invariance under the Cartan involution of G, whence the direct sum decomposition of $\mathfrak{h}$. Besides $\mathfrak{h}$ contains $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ by Lemma 1 and, for $V \in \mathfrak{h} \cap \mathfrak{p}$, the point $\exp V \cdot x_o = \mathrm{Exp}\, V$ belongs to $H \cdot x_o = \mathrm{Exp}\, \mathfrak{s}$, thus $V \in \mathfrak{s}$ by the injectivity of Exp on $\mathfrak{p}$.

By Lemma 1 the Radon transform of $u \in C_c(X)$ is given by

$$Ru(y) = \int_y u(x)\, dm_y(x) = \int_{\mathrm{Exp}\, \mathfrak{s}} u(g \cdot x)\, dm_{y_o}(x) ,$$

where dm_{y_o} is the Riemannian measure induced by X on its submanifold $y_o = \mathrm{Exp}\, \mathfrak{s}$.

b. Rank one case. We now restrict to the rank one case (hyperbolic spaces). Let $H \in \mathfrak{s}$ be a fixed non zero vector. The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{s}$, and $\mathrm{Exp}\, \mathfrak{s}$ is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of $(\mathrm{ad}\, H)^2$, with respective eigenvalues 0, $(\alpha(H))^2$, $(2\alpha(H))^2$ (where α and 2α are the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$), implies a similar decomposition of the invariant subspace $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha}\,,$$

with $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$ and $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$. We set

$$p = \dim \mathfrak{p}_\alpha\,, \quad q = \dim \mathfrak{p}_{2\alpha}\,, \quad n = \dim X = p + q + 1\,,$$
$$p' = \dim \mathfrak{s}_\alpha\,, \quad q' = \dim \mathfrak{s}_{2\alpha}\,, \quad d = \dim \mathfrak{s} = p' + q' + 1\,,$$

with $q = q' = 0$ when 2α is not a root (case of real hyperbolic spaces).

Let us normalize the vector H by the condition $\alpha(H) = 1$. Multiplying if necessary the Riemannian metric of X by a constant factor, we may assume that the corresponding Euclidean norm on $\mathfrak{p}$ satisfies $\|H\| = 1$. Since Exp is a diffeomorphism of $\mathfrak{p}$ onto X, the integral of a function $u \in C_c(X)$ can be computed as

$$\int_X u(x)\,dx = \int_{\mathfrak{p}} u(\operatorname{Exp} Z)\,J(Z)\,dZ\,,$$

where $J(Z) = \det_{\mathfrak{p}}(\sinh \operatorname{ad} Z / \operatorname{ad} Z)$ is the Jacobian of Exp, a K-invariant function on $\mathfrak{p}$. If u is K-invariant on X, we simply write $u(r)$ for $u(\operatorname{Exp} Z) = u(\operatorname{Exp} rH)$ with $r = \|Z\|$ whence, computing with spherical coordinates on $\mathfrak{p}$,

$$\int_X u(x)\,dx = \int_0^\infty u(r)A(r)\,dr\,,$$

where $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \operatorname{ad} rH / \operatorname{ad} rH)$ is the area of the sphere with center x_o and radius r in X, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$. Taking account of the eigenvalues of $(\operatorname{ad} H)^2$ we obtain, with a parameter ε explained in the next remark,

$$(4) \quad A(r) = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^p \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^q = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{n-1} (\cosh \varepsilon r)^q\,.$$

A similar expression gives $A_o(r)$ for the submanifold y_o (with d, p', q' instead of n, p, q). The distribution S in Proposition 4 is thus defined by the radial function

$$(5) \qquad S(r) = A_o(r)/A(r) = \left(\omega_d/\omega_n\right) \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{d-n} (\cosh \varepsilon r)^{q'-q}\,.$$

REMARK. Here $\varepsilon = 1$ for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting $\varepsilon = 0$ and $(\sinh \varepsilon r)/\varepsilon = r$: when $X = \mathbf{R}^n$ the geodesic submanifolds are the affine d-planes, $1 \le d \le n-1$, and

$$S(r) = \left(\omega_d/\omega_n\right) r^{d-n}\,.$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize H by $\alpha(H) = i$ and replace ε by i. Integrals with respect to r should run from 0 to the diameter ℓ of X, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

4.2 An inversion formula

The G-invariant differential operators on an isotropic space X are the polynomials of its Laplace-Beltrami operator L ([9], p. 288). In order to invert the d-geodesic Radon transform on X, Section 3.2 suggests looking for a polynomial P such that the above distribution S is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on X defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^b ,$$

where a and b are real constants and r is the distance from the origin x_o; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q} , \quad S(r) = \left(\omega_d / \omega_n\right) f_{d-n,q'-q}$$

with q, q', n and d as defined above.

Proposition 6. *Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a K-invariant distribution $F_{2k-n,b}$ on X such that*

$$P_k(L) F_{2k-n,b} = \omega_n 2^{k-1} (k-1)! \, (2-n)(4-n) \cdots (2k-n) \delta ,$$

where δ is the Dirac distribution at the origin x_o and P_k is the polynomial

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2 (n - 2j - b)(2j + b + q - 1)\right) .$$

Remark. The case $b = 0$, $n = 2k + 2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.