

# 4.1 Totally geodesic submanifolds

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

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$$u = (R^*Ru) * T.$$

Though the question can be tackled by harmonic analysis on  $X$  (cf. Section 5), a  $G$ -invariant linear differential operator  $D$  can sometimes be found directly, such that  $DS = \delta$ . Then (2) follows from the equality  $u = u * DS = D(u * S)$ . Indeed, for any test function  $\varphi$ ,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \langle u(g \cdot x_0), \langle S, ({}^t D\varphi) \circ \tau(g) \rangle \rangle \quad \text{by (1)} \\ &= \langle u(g \cdot x_0), \langle S, {}^t D(\varphi \circ \tau(g)) \rangle \rangle, \end{aligned}$$

since the transpose operator  ${}^t D$  is  $G$ -invariant too, as follows from the existence of a  $G$ -invariant measure on  $X$ . Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u(g \cdot x_0), \langle DS, \varphi \circ \tau(g) \rangle \rangle \\ &= \langle u * DS, \varphi \rangle, \end{aligned}$$

as claimed; assuming  $G$  unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can find a  $G$ -invariant differential operator  $D$  on  $X$  with given fundamental solution  $S$ . We shall now investigate this question on the basis of Propositions 4 and 5.

#### 4. RADON TRANSFORMS ON ISOTROPIC SPACES

Throughout this section  $X$  will be an isotropic connected noncompact Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}), H^{2m}(\mathbf{C}), H^{4m}(\mathbf{H}), H^{16}(\mathbf{O}),$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the  $d$ -geodesic Radon transform on  $X$ , defined by integrating over a family of  $d$ -dimensional totally geodesic submanifolds of  $X$ . At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on  $H^{2k+1}(\mathbf{R})$ .

##### 4.1 TOTALLY GEODESIC SUBMANIFOLDS

Our first goal is to describe these submanifolds and the corresponding functions  $S$  in Proposition 4.

**a.** Let  $X = G/K$  be a Riemannian symmetric space of the noncompact type (of arbitrary rank), where  $G$  is a connected semisimple Lie group and  $K$  a maximal compact subgroup (see Notations, **c** and **d**).

At the Lie algebra level, a totally geodesic submanifold of  $X$  is defined by a *Lie triple system*, i.e. a vector subspace  $\mathfrak{s}$  of  $\mathfrak{p}$  such that  $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$ . Then  $\text{Exp } \mathfrak{s}$  is totally geodesic in  $X$  and contains the origin  $x_o$ . Besides  $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$  and  $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$  are Lie subalgebras of  $\mathfrak{g}$ . Let  $G'$  be the (closed) Lie subgroup with Lie algebra  $\mathfrak{g}'$ , and  $K'$  (with Lie algebra  $\mathfrak{k}'$ ) be the isotropy subgroup of  $x_o$  in  $G'$ . Then

$$\text{Exp } \mathfrak{s} = G'/K' = G' \cdot x_o,$$

a closed symmetric subspace of  $X$  ([8], p.224–226, or [15], p.234 sq.).

Now let  $Y$  be the set of all  $d$ -dimensional totally geodesic submanifolds  $y = g \cdot y_o$  of  $X$ , with  $g \in G$  and  $y_o = \text{Exp } \mathfrak{s} = G' \cdot x_o$ . *Lemma 1* applies: if  $H$  is the subgroup of all  $h \in G$  such that  $h \cdot y_o = y_o$ , then  $y_o = H \cdot x_o$ ,  $Y = G/H$  and the incidence relation is  $x \in y$ .

It will be useful to note that the Lie algebra  $\mathfrak{h}$  of  $H$  satisfies

$$(3) \quad \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}), \quad \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s}, \mathfrak{s}], \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s}.$$

Indeed the definition of  $H$  shows its invariance under the Cartan involution of  $G$ , whence the direct sum decomposition of  $\mathfrak{h}$ . Besides  $\mathfrak{h}$  contains  $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$  by Lemma 1 and, for  $V \in \mathfrak{h} \cap \mathfrak{p}$ , the point  $\exp V \cdot x_o = \text{Exp } V$  belongs to  $H \cdot x_o = \text{Exp } \mathfrak{s}$ , thus  $V \in \mathfrak{s}$  by the injectivity of  $\text{Exp}$  on  $\mathfrak{p}$ .

By Lemma 1 the Radon transform of  $u \in C_c(X)$  is given by

$$Ru(y) = \int_y u(x) dm_y(x) = \int_{\text{Exp } \mathfrak{s}} u(g \cdot x) dm_{y_o}(x),$$

where  $dm_{y_o}$  is the Riemannian measure induced by  $X$  on its submanifold  $y_o = \text{Exp } \mathfrak{s}$ .

**b.** RANK ONE CASE. We now restrict to the rank one case (hyperbolic spaces). Let  $H \in \mathfrak{s}$  be a fixed non zero vector. The line  $\mathfrak{a} = \mathbf{R}H$  is a maximal abelian subspace of  $\mathfrak{p}$  and  $\mathfrak{s}$ , and  $\text{Exp } \mathfrak{s}$  is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of  $(\text{ad } H)^2$ , with respective eigenvalues  $0$ ,  $(\alpha(H))^2$ ,  $(2\alpha(H))^2$  (where  $\alpha$  and  $2\alpha$  are the positive roots of the pair  $(\mathfrak{g}, \mathfrak{a})$ ), implies a similar decomposition of the invariant subspace  $\mathfrak{s}$ :

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha},$$

with  $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$  and  $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$ . We set

$$\begin{aligned} p &= \dim \mathfrak{p}_\alpha, & q &= \dim \mathfrak{p}_{2\alpha}, & n &= \dim X = p + q + 1, \\ p' &= \dim \mathfrak{s}_\alpha, & q' &= \dim \mathfrak{s}_{2\alpha}, & d &= \dim \mathfrak{s} = p' + q' + 1, \end{aligned}$$

with  $q = q' = 0$  when  $2\alpha$  is not a root (case of real hyperbolic spaces).

Let us normalize the vector  $H$  by the condition  $\alpha(H) = 1$ . Multiplying if necessary the Riemannian metric of  $X$  by a constant factor, we may assume that the corresponding Euclidean norm on  $\mathfrak{p}$  satisfies  $\|H\| = 1$ . Since  $\text{Exp}$  is a diffeomorphism of  $\mathfrak{p}$  onto  $X$ , the integral of a function  $u \in C_c(X)$  can be computed as

$$\int_X u(x) dx = \int_{\mathfrak{p}} u(\text{Exp } Z) J(Z) dZ,$$

where  $J(Z) = \det_{\mathfrak{p}}(\sinh \text{ad } Z / \text{ad } Z)$  is the Jacobian of  $\text{Exp}$ , a  $K$ -invariant function on  $\mathfrak{p}$ . If  $u$  is  $K$ -invariant on  $X$ , we simply write  $u(r)$  for  $u(\text{Exp } Z) = u(\text{Exp } rH)$  with  $r = \|Z\|$  whence, computing with spherical coordinates on  $\mathfrak{p}$ ,

$$\int_X u(x) dx = \int_0^\infty u(r) A(r) dr,$$

where  $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \text{ad } rH / \text{ad } rH)$  is the area of the sphere with center  $x_o$  and radius  $r$  in  $X$ , and  $\omega_n = 2\pi^{n/2} / \Gamma(n/2)$  is the area of the unit sphere in  $\mathbf{R}^n$ . Taking account of the eigenvalues of  $(\text{ad } H)^2$  we obtain, with a parameter  $\varepsilon$  explained in the next remark,

$$(4) \quad A(r) = \omega_n \left( \frac{\sinh \varepsilon r}{\varepsilon} \right)^p \left( \frac{\sinh 2\varepsilon r}{2\varepsilon} \right)^q = \omega_n \left( \frac{\sinh \varepsilon r}{\varepsilon} \right)^{n-1} (\cosh \varepsilon r)^q.$$

A similar expression gives  $A_o(r)$  for the submanifold  $y_o$  (with  $d, p', q'$  instead of  $n, p, q$ ). The distribution  $S$  in Proposition 4 is thus defined by the radial function

$$(5) \quad S(r) = A_o(r)/A(r) = (\omega_d/\omega_n) \left( \frac{\sinh \varepsilon r}{\varepsilon} \right)^{d-n} (\cosh \varepsilon r)^{q'-q}.$$

REMARK. Here  $\varepsilon = 1$  for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting  $\varepsilon = 0$  and  $(\sinh \varepsilon r)/\varepsilon = r$ : when  $X = \mathbf{R}^n$  the geodesic submanifolds are the affine  $d$ -planes,  $1 \leq d \leq n - 1$ , and

$$S(r) = (\omega_d/\omega_n) r^{d-n}.$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize  $H$  by  $\alpha(H) = i$  and replace  $\varepsilon$  by  $i$ . Integrals with respect to  $r$  should run from 0 to the diameter  $\ell$  of  $X$ , i.e. the first number  $\ell > 0$  such that  $A(\ell) = 0$ .

## 4.2 AN INVERSION FORMULA

The  $G$ -invariant differential operators on an isotropic space  $X$  are the polynomials of its Laplace-Beltrami operator  $L$  ([9], p. 288). In order to invert the  $d$ -geodesic Radon transform on  $X$ , Section 3.2 suggests looking for a polynomial  $P$  such that the above distribution  $S$  is a fundamental solution of  $P(L)$ .

Motivated by (4) and (5), we introduce the family of radial functions  $f_{a,b}$  on  $X$  defined by

$$f_{a,b}(r) = \left( \frac{\sinh \varepsilon r}{\varepsilon} \right)^a (\cosh \varepsilon r)^b = \left( \frac{\sinh \varepsilon r}{\varepsilon} \right)^{a-b} \left( \frac{\sinh 2\varepsilon r}{2\varepsilon} \right)^b,$$

where  $a$  and  $b$  are real constants and  $r$  is the distance from the origin  $x_o$ ; in particular  $f_{a,b}(r) = r^a$  for  $\varepsilon = 0$ . Thus

$$A(r) = \omega_n f_{n-1,q}, \quad S(r) = (\omega_d/\omega_n) f_{d-n,q'-q}$$

with  $q$ ,  $q'$ ,  $n$  and  $d$  as defined above.

**PROPOSITION 6.** *Assume  $\varepsilon = 0$  (Euclidean case), or  $\varepsilon = 1$  and  $b = 0$ , or else  $\varepsilon = 1$  and  $b = 1 - q$  (hyperbolic cases). Then, for any integer  $k \geq 1$ , the function  $f_{2k-n,b}$  defines a  $K$ -invariant distribution  $F_{2k-n,b}$  on  $X$  such that*

$$P_k(L)F_{2k-n,b} = \omega_n 2^{k-1} (k-1)! (2-n)(4-n) \cdots (2k-n) \delta,$$

where  $\delta$  is the Dirac distribution at the origin  $x_o$  and  $P_k$  is the polynomial

$$P_k(x) = \prod_{j=1}^k (x + \varepsilon^2(n - 2j - b)(2j + b + q - 1)).$$

**REMARK.** The case  $b = 0$ ,  $n = 2k + 2$  was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.