$$u = (R^* R u) * T\,.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G-invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ,

$$\begin{aligned} \langle D(u*S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \big\langle u(g\cdot x_o), \langle S, ({}^t D\varphi)\circ \tau(g)\rangle \big\rangle \qquad \text{by (1)} \\ &= \big\langle u(g\cdot x_o), \langle S, {}^t D(\varphi \circ \tau(g))\rangle \big\rangle\,, \end{aligned}$$

since the transpose operator ${}^t D$ is G-invariant too, as follows from the existence of a G-invariant measure on X. Finally,

$$\begin{aligned} \langle D(u*S), \varphi \rangle &= \big\langle u(g\cdot x_o), \langle DS, \varphi\circ\tau(g)\rangle \big\rangle \\ &= \langle u * DS, \varphi \rangle\,, \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can *find a G-invariant differential operator D on X with given fundamental solution S*. We shall now investigate this question on the basis of Propositions 4 and 5.

4. Radon transforms on isotropic spaces

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}),\ H^{2m}(\mathbf{C}),\ H^{4m}(\mathbf{H}),\ H^{16}(\mathbf{O})\,,$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the d-geodesic Radon transform on X, defined by integrating over a family of d-dimensional totally geodesic submanifolds of X. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 Totally geodesic submanifolds

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.

a. Let $X = G/K$ be a *Riemannian symmetric space of the noncompact type (of arbitrary rank)*, where G is a connected semisimple Lie group and K a maximal compact subgroup (see Notations, **c** and **d**).

At the Lie algebra level, a totally geodesic submanifold of X is defined by a *Lie triple system*, i.e. a vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ such that $[\mathfrak{s},[\mathfrak{s},\mathfrak{s}]] \subset \mathfrak{s}$. Then $\operatorname{Exp} \mathfrak{s}$ is totally geodesic in X and contains the origin x_o. Besides $\mathfrak{k}' = [\mathfrak{s},\mathfrak{s}] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$ are Lie subalgebras of $\mathfrak{g}$. Let G' be the (closed) Lie subgroup with Lie algebra $\mathfrak{g}'$, and K' (with Lie algebra $\mathfrak{k}'$) be the isotropy subgroup of x_o in G'. Then

$$\operatorname{Exp} \mathfrak{s} = G'/K' = G' \cdot x_o\,,$$

a closed symmetric subspace of X ([8], p. 224–226, or [15], p. 234 sq.).

Now let Y be the set of all d-dimensional totally geodesic submanifolds $y = g \cdot y_o$ of X, with $g \in G$ and $y_o = \operatorname{Exp} \mathfrak{s} = G' \cdot x_o$. *Lemma 1 applies*: if H is the subgroup of all $h \in G$ such that $h \cdot y_o = y_o$, then $y_o = H \cdot x_o$, $Y = G/H$ and the incidence relation is $x \in y$.

It will be useful to note that the Lie algebra $\mathfrak{h}$ of H satisfies

$$\text{(3)} \qquad \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})\,, \quad \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s},\mathfrak{s}]\,, \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s}\,.$$

Indeed the definition of H shows its invariance under the Cartan involution of G, whence the direct sum decomposition of $\mathfrak{h}$. Besides $\mathfrak{h}$ contains $\mathfrak{g}' = [\mathfrak{s},\mathfrak{s}] \oplus \mathfrak{s}$ by Lemma 1 and, for $V \in \mathfrak{h} \cap \mathfrak{p}$, the point $\exp V \cdot x_o = \operatorname{Exp} V$ belongs to $H \cdot x_o = \operatorname{Exp} \mathfrak{s}$, thus $V \in \mathfrak{s}$ by the injectivity of Exp on $\mathfrak{p}$.

By Lemma 1 the Radon transform of $u \in C_c(X)$ is given by

$$Ru(y) = \int_y u(x)\, dm_y(x) = \int_{\operatorname{Exp} \mathfrak{s}} u(g \cdot x)\, dm_{y_o}(x)\,,$$

where dm_{y_o} is the Riemannian measure induced by X on its submanifold $y_o = \operatorname{Exp} \mathfrak{s}$.

b. Rank one case. We now restrict to the rank one case (hyperbolic spaces). Let $H \in \mathfrak{s}$ be a fixed non zero vector. The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{s}$, and $\operatorname{Exp} \mathfrak{s}$ is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of $(\operatorname{ad} H)^2$, with respective eigenvalues 0, $(\alpha(H))^2$, $(2\alpha(H))^2$ (where α and 2α are the positive roots of the pair $(\mathfrak{g},\mathfrak{a})$), implies a similar decomposition of the invariant subspace $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha}\,,$$

with $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$ and $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$. We set

$$p = \dim \mathfrak{p}_\alpha\,, \quad q = \dim \mathfrak{p}_{2\alpha}\,, \quad n = \dim X = p + q + 1\,,$$
$$p' = \dim \mathfrak{s}_\alpha\,, \quad q' = \dim \mathfrak{s}_{2\alpha}\,, \quad d = \dim \mathfrak{s} = p' + q' + 1\,,$$

with $q = q' = 0$ when 2α is not a root (case of real hyperbolic spaces).

Let us normalize the vector H by the condition $\alpha(H) = 1$. Multiplying if necessary the Riemannian metric of X by a constant factor, we may assume that the corresponding Euclidean norm on $\mathfrak{p}$ satisfies $\|H\| = 1$. Since Exp is a diffeomorphism of $\mathfrak{p}$ onto X, the integral of a function $u \in C_c(X)$ can be computed as

$$\int_X u(x)\, dx = \int_{\mathfrak{p}} u(\mathrm{Exp}\ Z)\, J(Z)\, dZ\,,$$

where $J(Z) = \det_{\mathfrak{p}}(\sinh \operatorname{ad} Z / \operatorname{ad} Z)$ is the Jacobian of Exp, a K-invariant function on $\mathfrak{p}$. If u is K-invariant on X, we simply write $u(r)$ for $u(\mathrm{Exp}\ Z) = u(\mathrm{Exp}\ rH)$ with $r = \|Z\|$ whence, computing with spherical coordinates on $\mathfrak{p}$,

$$\int_X u(x)\, dx = \int_0^\infty u(r) A(r)\, dr\,,$$

where $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \operatorname{ad} rH / \operatorname{ad} rH)$ is the area of the sphere with center x_o and radius r in X, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$. Taking account of the eigenvalues of $(\operatorname{ad} H)^2$ we obtain, with a parameter ε explained in the next remark,

$$(4) \quad A(r) = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^p \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^q = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{n-1} (\cosh \varepsilon r)^q\,.$$

A similar expression gives $A_o(r)$ for the submanifold y_o (with d, p', q' instead of n, p, q). The distribution S in Proposition 4 is thus defined by the radial function

$$(5) \qquad S(r) = A_o(r)/A(r) = \left(\omega_d/\omega_n\right) \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{d-n} (\cosh \varepsilon r)^{q'-q}\,.$$

REMARK. Here $\varepsilon = 1$ for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting $\varepsilon = 0$ and $(\sinh \varepsilon r)/\varepsilon = r$: when $X = \mathbf{R}^n$ the geodesic submanifolds are the affine d-planes, $1 \le d \le n-1$, and

$$S(r) = \left(\omega_d/\omega_n\right) r^{d-n}\,.$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize H by $\alpha(H) = i$ and replace ε by i. Integrals with respect to r should run from 0 to the diameter ℓ of X, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

4.2 AN INVERSION FORMULA

The G-invariant differential operators on an isotropic space X are the polynomials of its Laplace-Beltrami operator L ([9], p. 288). In order to invert the d-geodesic Radon transform on X, Section 3.2 suggests looking for a polynomial P such that the above distribution S is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on X defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^b ,$$

where a and b are real constants and r is the distance from the origin x_o; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q} , \quad S(r) = \left(\omega_d/\omega_n\right) f_{d-n,q'-q}$$

with q, q', n and d as defined above.

PROPOSITION 6. *Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a K-invariant distribution $F_{2k-n,b}$ on X such that*

$$P_k(L)F_{2k-n,b} = \omega_n 2^{k-1}(k-1)!\,(2-n)(4-n)\cdots(2k-n)\,\delta ,$$

where δ is the Dirac distribution at the origin x_o and P_k is the polynomial

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2(n-2j-b)(2j+b+q-1)\right) .$$

REMARK. The case $b = 0$, $n = 2k+2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.

Proof. By [9], p. 313 the radial part of L is

$$\Delta = \partial_r^2 + \frac{A'(r)}{A(r)}\partial_r = A(r)^{-1} \circ \partial_r \circ A(r) \circ \partial_r$$
$$= \partial_r^2 + ((n-1)\varepsilon \coth \varepsilon r + q\varepsilon \tanh \varepsilon r)\, \partial_r$$
$$= \partial_r^2 + (p\varepsilon \coth \varepsilon r + 2q\varepsilon \coth 2\varepsilon r)\, \partial_r\, .$$

The proof of the proposition breaks up into a few easy facts. First we have, for any $a, b \in \mathbf{R}$, the following equality of functions of $r > 0$:

$$(6)\quad \left(\Delta - \varepsilon^2(a+b)(a+n+b+q-1)\right) f_{a,b} = a(a+n-2)f_{a-2,b} - \varepsilon^2 b(b+q-1)f_{a,b-2}\, ,$$

which is immediate from $\Delta f = A^{-1}(Af')'$ and the identities

$$f'_{a,b} = af_{a-1,b+1} + \varepsilon^2 bf_{a+1,b-1}\, , \quad f_{a,b} = f_{a,b-2} + \varepsilon^2 f_{a+2,b-2}\, .$$

LEMMA 7. *For $a+n \geq 2$, $\varepsilon = 0$ or 1, the locally integrable function $f_{a,b}$ defines a K-invariant distribution $F_{a,b}$ on X such that*

$$\left(L - \varepsilon^2(a+b)(a+n+b+q-1)\right) F_{a,b} = \begin{cases} a(a+n-2)F_{a-2,b} - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n > 2 \\ \omega_n a\, \delta - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n = 2 \end{cases}$$

(equality of distributions on X).

EXAMPLE. Taking $b = 0$, resp. $b = 1-q$, the lemma provides the following fundamental solutions (which coincide for $q = 1$)

$$\left(L + \varepsilon^2(n-2)(q+1)\right) \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n} = \omega_n(2-n)\delta$$
$$\left(L + 2\varepsilon^2(n+q-3)\right) \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n} (\cosh \varepsilon r)^{1-q} = \omega_n(2-n)\delta\, .$$

In the flat case $\varepsilon = 0$ they both reduce to $Lr^{2-n} = \omega_n(2-n)\delta$, a classical result for $\mathbf{R}^n$.

Proof of Lemma 7. Due to the K-invariance of $f_{a,b}$ and L we need only consider K-invariant test functions $u \in \mathcal{D}(X)$. The integral

$$\int_X f_{a,b} \cdot u\, dx = \int_0^\infty f_{a,b}(r)u(r)A(r)\, dr = \omega_n \int_0^\infty f_{a+n-1,b+q}(r)u(r)\, dr\, ,$$

absolutely convergent if $a+n>0$, defines a distribution $F_{a,b}$ on X. In view of the symmetry and K-invariance of the Laplace operator we have

$$\begin{aligned}\langle LF_{a,b},u\rangle &= \langle F_{a,b},Lu\rangle\\ &= \int_0^\infty f_{a,b}(r)\Delta u(r)A(r)\,dr = \int_0^\infty f_{a,b}(Au')'\,dr\\ &= \left(Af'_{a,b}u\right)(0) - \left(Af_{a,b}u'\right)(0) + \int_0^\infty (Af'_{a,b})'u\,dr\,.\end{aligned}$$

If $a+n>2$ the function $Af_{a,b}$ vanishes to order $a+n-1$ at the origin, and $Af'_{a,b}$ to order $a+n-2$. Since $u(r)$ is smooth (this notation stands for $u(\mathrm{Exp}\, rH)$ with $\|H\|=1$), it follows that

$$\langle LF_{a,b},u\rangle = \int_0^\infty \Delta f_{a,b}(r)u(r)A(r)\,dr\,,$$

whence the result by (6).

The case $a+n=2$ is similar, in view of $(Af'_{a,b})(0)=\omega_n a$. □

Proposition 6 now follows easily: letting

$$L_a = L - \varepsilon^2(a+b)(a+n+b+q-1)$$

we have, by Lemma 7,

$$L_aF_{a,b} = \begin{cases} a(a+n-2)\,F_{a-2,b} & \text{if } a+n>2\\ \omega_n a\,\delta & \text{if } a+n=2\end{cases}$$

whenever $\varepsilon^2 b(b+q-1)=0$. Since

$$P_k(L) = L_{2-n}L_{4-n}\cdots L_{2k-n}\,,$$

the proposition follows by induction on k. □

THEOREM 8. *The d-geodesic Radon transform on a n-dimensional non-compact Riemannian isotropic space X can be inverted by means of a polynomial of its Laplace-Beltrami operator L, under the following assumptions:*

(i) *d is even: $d=2k$ with $k\geq 1$;*

(ii) *$X=\mathbf{R}^n$, or $\dim\mathfrak{s}_{2\alpha}=\dim\mathfrak{p}_{2\alpha}$, or else $\dim\mathfrak{s}_{2\alpha}=1$.*

Then

$$Cu = P_k(L)R^*Ru\,,$$

for any $u\in\mathcal{D}(X)$, where P_k is the polynomial from Proposition 6 (with $\varepsilon=1$, $q=\dim\mathfrak{p}_{2\alpha}$ and $b+q=\dim\mathfrak{s}_{2\alpha}$ if X is hyperbolic, or $\varepsilon=0$ if $X=\mathbf{R}^n$) and

$$C = \omega_d(-1)^k 2^{k-1}(k-1)!(n-2)(n-4)\cdots(n-2k)\,.$$

Proof. By (5) one has $S = (\omega_d/\omega_n) f_{a,b}$, with $a = d - n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (Section 4.1 **b**). The theorem follows from Proposition 6 and Section 3.2. □

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9], Chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial P_k is related to representation theory.

4.3 Examples

According to assumption (ii), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbf{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space Y consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbf{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

a. A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8], p. 335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is 2, 4 or 8; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

b. Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace H by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbf{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}, \quad \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}.$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7], p. 171, or [9], p. 168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

c. Totally geodesic transform on classical hyperbolic spaces. Let $X = H^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1, m; \mathbf{F})$, $K = U(1; \mathbf{F}) \times U(m; \mathbf{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices

$$V = \begin{pmatrix} 0 & \overline{V}_1 & \cdots & \overline{V}_m \\ V_1 & & & \\ \vdots & & (0) & \\ V_m & & & \end{pmatrix}, \quad V_i \in \mathbf{F},$$

can be identified with $\mathbf{F}^m$.

Let $\overline{V} \cdot W = \sum_{i=1}^m \overline{V}_i W_i$. For $U, V, W \in \mathfrak{p} = \mathbf{F}^m$, easy computations lead to

$$[U,[V,W]] = U\left(\overline{V} \cdot W - \overline{W} \cdot V\right) - V(\overline{W} \cdot U) + W(\overline{V} \cdot U) \tag{7}$$

($\mathbf{F}^m$ being considered as a $\mathbf{F}$-vector space, with scalars acting on the right). It follows that *any* $\mathbf{F}$*-subspace* $\mathfrak{s}$ *of* $\mathfrak{p}$ *is a Lie triple system.* Similarly, the natural inclusions $\mathbf{R}^m \subset \mathbf{C}^m \subset \mathbf{H}^m$ show that any $\mathbf{R}$-subspace of $\mathfrak{p} \cap \mathbf{R}^m$, or any $\mathbf{C}$-subspace of $\mathfrak{p} \cap \mathbf{C}^m$, is a Lie triple system.

Let $H \neq 0$ be an element of $\mathfrak{p}$. The eigenspaces of $(\operatorname{ad} H)^2$ can be obtained from (7), whence the decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}, \quad \mathfrak{a} = \mathbf{R}H,$$
$$\mathfrak{p}_\alpha = \{V \in \mathfrak{p} \mid \overline{H} \cdot V = 0\}, \quad \mathfrak{p}_{2\alpha} = \{H\lambda \mid \lambda \in \mathbf{F},\ \lambda + \overline{\lambda} = 0\},$$

with respective eigenvalues 0, $\overline{H} \cdot H$ and $4\left(\overline{H} \cdot H\right)$. A similar decomposition holds for $\mathfrak{s}$, if H is chosen in $\mathfrak{s}$. The spaces $\mathfrak{a} \oplus \mathfrak{p}_{2\alpha} = H\mathbf{F}$ and $\mathfrak{p}_\alpha$ are $\mathbf{F}$-subspaces of $\mathfrak{p}$, therefore Lie triple systems (as mentioned in a and b above). More generally, Theorem 8 applies to the following four families of totally geodesic submanifolds $\operatorname{Exp} \mathfrak{s}$; all superscripts in the table are real dimensions, with k, l, m strictly positive integers.

X	$\dim \mathfrak{p}_\alpha$	$\dim \mathfrak{p}_{2\alpha}$	$\mathfrak{s}$	$\dim \mathfrak{s}_\alpha$	$\dim \mathfrak{s}_{2\alpha}$	$y_o = \operatorname{Exp} \mathfrak{s}$
$H^m(\mathbf{R})$	$m-1$	0	(1)	$2k-1$	0	$H^{2k}(\mathbf{R})$
$H^{2m}(\mathbf{C})$	$2m-2$	1	(2)	$2k-2$	1	$H^{2k}(\mathbf{C})$
$H^{4m}(\mathbf{H})$	$4m-4$	3	(3)	$2k-2$	1	$H^{2k}(\mathbf{C})$
$H^{4m}(\mathbf{H})$	$4m-4$	3	(4)	$4l-4$	3	$H^{4l}(\mathbf{H})$

Case (1): $\mathfrak{s}$ is any $\mathbf{R}$-subspace of $\mathfrak{p} = \mathbf{R}^m$, with $\dim_{\mathbf{R}} \mathfrak{s} = 2k \leq m$.
Case (2): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathfrak{p} = \mathbf{C}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (3): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathbf{C}^m \subset \mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (4): $\mathfrak{s}$ is any $\mathbf{H}$-subspace of $\mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{H}} \mathfrak{s} = l \leq m$.

d. HOROCYCLE TRANSFORM ON REAL HYPERBOLIC SPACES. Proposition 6 also applies to this case, because of the similarity between the functions S obtained in Propositions 4 and 5.

Following the same steps as for geodesic submanifolds, one can find a polynomial of the Laplacian with fundamental solution S (case $q = 0$ in Proposition 5). Indeed $S(r)$ is now, up to a constant factor, $f_{-1,2-n}(r/2)$ in the notation of Section 4.2 with $\varepsilon = 1$. Let

$$\Delta_{p,q} = \partial_r^2 + (p \coth r + 2q \coth 2r)\, \partial_r$$

be the radial part of the Laplacian and $g(r) = f(r/2)$. Then

$$4\left(\Delta_{p,0}\, g\right)(r) = \left(\Delta_{0,p} f\right)\left(r/2\right) ;$$

note that the roles of p and q have been interchanged. The next theorem now follows from Propositions 5 and 6, with $n = 2k+1$, $\varepsilon = 1$ and $b = 1 - p = 2 - n$.

THEOREM 9 (Helgason). *The horocycle Radon transform on the odd-dimensional hyperbolic space $X = H^{2k+1}(\mathbf{R})$, $k \geq 1$, is inverted by*

$$Cu = Q_k(L) R^* R u\,,$$

where $u \in \mathcal{D}(X)$, L is the Laplace-Beltrami operator of X,

$$C = \left(-\frac{\pi}{2}\right)^k \frac{(2k-1)!}{(k-1)!}\,, \quad Q_k(x) = \prod_{j=1}^{k} (x + j(2k - j))\,.$$

In [11], p. 210, the normalization of the Riemannian metric on X differs from ours.

The result extends to the horocycle transform on a Riemannian symmetric space $X = G/K$ of the noncompact type, provided that the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below). The spaces $H^{2k+1}(\mathbf{R})$ in Theorem 9 are the rank one spaces among those.