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**Artikel:** INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

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$$u = (R^* R u) * T \, .$$

Though the question can be tackled by harmonic analysis on $X$ (cf. Section 5), a $G$-invariant linear differential operator $D$ can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function $\varphi$,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \big\langle u(g \cdot x_o), \langle S, ({}^t D\varphi) \circ \tau(g) \rangle \big\rangle \qquad \text{by (1)} \\ &= \big\langle u(g \cdot x_o), \langle S, {}^t D(\varphi \circ \tau(g)) \rangle \big\rangle \, , \end{aligned}$$

since the transpose operator ${}^t D$ is $G$-invariant too, as follows from the existence of a $G$-invariant measure on $X$. Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \big\langle u(g \cdot x_o), \langle DS, \varphi \circ \tau(g) \rangle \big\rangle \\ &= \langle u * DS, \varphi \rangle \, , \end{aligned}$$

as claimed; assuming $G$ unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can *find a $G$-invariant differential operator $D$ on $X$ with given fundamental solution $S$*. We shall now investigate this question on the basis of Propositions 4 and 5.

## 4. Radon transforms on isotropic spaces

Throughout this section $X$ will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}),\ H^{2m}(\mathbf{C}),\ H^{4m}(\mathbf{H}),\ H^{16}(\mathbf{O}) \, ,$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the $d$-geodesic Radon transform on $X$, defined by integrating over a family of $d$-dimensional totally geodesic submanifolds of $X$. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

### 4.1 Totally geodesic submanifolds

Our first goal is to describe these submanifolds and the corresponding functions $S$ in Proposition 4.

**a.** Let $X = G/K$ be a *Riemannian symmetric space of the noncompact type (of arbitrary rank)*, where $G$ is a connected semisimple Lie group and $K$ a maximal compact subgroup (see Notations, **c** and **d**).

At the Lie algebra level, a totally geodesic submanifold of $X$ is defined by a *Lie triple system*, i.e. a vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ such that $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Then $\operatorname{Exp} \mathfrak{s}$ is totally geodesic in $X$ and contains the origin $x_o$. Besides $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$ are Lie subalgebras of $\mathfrak{g}$. Let $G'$ be the (closed) Lie subgroup with Lie algebra $\mathfrak{g}'$, and $K'$ (with Lie algebra $\mathfrak{k}'$) be the isotropy subgroup of $x_o$ in $G'$. Then

$$\operatorname{Exp} \mathfrak{s} = G'/K' = G' \cdot x_o \,,$$

a closed symmetric subspace of $X$ ([8], p. 224–226, or [15], p. 234 sq.).

Now let $Y$ be the set of all $d$-dimensional totally geodesic submanifolds $y = g \cdot y_o$ of $X$, with $g \in G$ and $y_o = \operatorname{Exp} \mathfrak{s} = G' \cdot x_o$. *Lemma 1 applies*: if $H$ is the subgroup of all $h \in G$ such that $h \cdot y_o = y_o$, then $y_o = H \cdot x_o$, $Y = G/H$ and the incidence relation is $x \in y$.

It will be useful to note that the Lie algebra $\mathfrak{h}$ of $H$ satisfies

$$\text{(3)} \qquad \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \,, \quad \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s}, \mathfrak{s}] \,, \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s} \,.$$

Indeed the definition of $H$ shows its invariance under the Cartan involution of $G$, whence the direct sum decomposition of $\mathfrak{h}$. Besides $\mathfrak{h}$ contains $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ by Lemma 1 and, for $V \in \mathfrak{h} \cap \mathfrak{p}$, the point $\exp V \cdot x_o = \operatorname{Exp} V$ belongs to $H \cdot x_o = \operatorname{Exp} \mathfrak{s}$, thus $V \in \mathfrak{s}$ by the injectivity of Exp on $\mathfrak{p}$.

By Lemma 1 the Radon transform of $u \in C_c(X)$ is given by

$$Ru(y) = \int_y u(x)\, dm_y(x) = \int_{\operatorname{Exp} \mathfrak{s}} u(g \cdot x)\, dm_{y_o}(x) \,,$$

where $dm_{y_o}$ is the Riemannian measure induced by $X$ on its submanifold $y_o = \operatorname{Exp} \mathfrak{s}$.

**b.** Rank one case. We now restrict to the rank one case (hyperbolic spaces). Let $H \in \mathfrak{s}$ be a fixed non zero vector. The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{s}$, and $\operatorname{Exp} \mathfrak{s}$ is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of $(\operatorname{ad} H)^2$, with respective eigenvalues $0$, $(\alpha(H))^2$, $(2\alpha(H))^2$ (where $\alpha$ and $2\alpha$ are the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$), implies a similar decomposition of the invariant subspace $\mathfrak{s}$ :

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha} ,$$

with $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$ and $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$. We set

$$p = \dim \mathfrak{p}_\alpha , \quad q = \dim \mathfrak{p}_{2\alpha} , \quad n = \dim X = p + q + 1 ,$$
$$p' = \dim \mathfrak{s}_\alpha , \quad q' = \dim \mathfrak{s}_{2\alpha} , \quad d = \dim \mathfrak{s} = p' + q' + 1 ,$$

with $q = q' = 0$ when $2\alpha$ is not a root (case of real hyperbolic spaces).

Let us normalize the vector $H$ by the condition $\alpha(H) = 1$. Multiplying if necessary the Riemannian metric of $X$ by a constant factor, we may assume that the corresponding Euclidean norm on $\mathfrak{p}$ satisfies $\|H\| = 1$. Since Exp is a diffeomorphism of $\mathfrak{p}$ onto $X$, the integral of a function $u \in C_c(X)$ can be computed as

$$\int_X u(x)\, dx = \int_{\mathfrak{p}} u(\mathrm{Exp}\ Z)\, J(Z)\, dZ ,$$

where $J(Z) = \det_{\mathfrak{p}}(\sinh \mathrm{ad}\, Z / \mathrm{ad}\, Z)$ is the Jacobian of Exp, a $K$-invariant function on $\mathfrak{p}$. If $u$ is $K$-invariant on $X$, we simply write $u(r)$ for $u(\mathrm{Exp}\, Z) = u(\mathrm{Exp}\, rH)$ with $r = \|Z\|$ whence, computing with spherical coordinates on $\mathfrak{p}$,

$$\int_X u(x)\, dx = \int_0^\infty u(r) A(r)\, dr ,$$

where $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \mathrm{ad}\, rH / \mathrm{ad}\, rH)$ is the area of the sphere with center $x_o$ and radius $r$ in $X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$. Taking account of the eigenvalues of $(\mathrm{ad}\, H)^2$ we obtain, with a parameter $\varepsilon$ explained in the next remark,

$$(4) \quad A(r) = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^p \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^q = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{n-1} (\cosh \varepsilon r)^q .$$

A similar expression gives $A_o(r)$ for the submanifold $y_o$ (with $d, p', q'$ instead of $n, p, q$). The distribution $S$ in Proposition 4 is thus defined by the radial function

$$(5) \qquad S(r) = A_o(r)/A(r) = \left(\omega_d/\omega_n\right) \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{d-n} (\cosh \varepsilon r)^{q'-q} .$$

REMARK. Here $\varepsilon = 1$ for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting $\varepsilon = 0$ and $(\sinh \varepsilon r)/\varepsilon = r$: when $X = \mathbf{R}^n$ the geodesic submanifolds are the affine $d$-planes, $1 \leq d \leq n-1$, and

$$S(r) = \left(\omega_d/\omega_n\right) r^{d-n} .$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize $H$ by $\alpha(H) = i$ and replace $\varepsilon$ by $i$. Integrals with respect to $r$ should run from 0 to the diameter $\ell$ of $X$, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

## 4.2 An inversion formula

The $G$-invariant differential operators on an isotropic space $X$ are the polynomials of its Laplace-Beltrami operator $L$ ([9], p. 288). In order to invert the $d$-geodesic Radon transform on $X$, Section 3.2 suggests looking for a polynomial $P$ such that the above distribution $S$ is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on $X$ defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^b ,$$

where $a$ and $b$ are real constants and $r$ is the distance from the origin $x_o$; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q} , \quad S(r) = \left(\omega_d/\omega_n\right) f_{d-n,q'-q}$$

with $q$, $q'$, $n$ and $d$ as defined above.

Proposition 6. *Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a $K$-invariant distribution $F_{2k-n,b}$ on $X$ such that*

$$P_k(L) F_{2k-n,b} = \omega_n 2^{k-1} (k-1)! \, (2-n)(4-n) \cdots (2k-n) \delta ,$$

*where $\delta$ is the Dirac distribution at the origin $x_o$ and $P_k$ is the polynomial*

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2 (n - 2j - b)(2j + b + q - 1)\right) .$$

Remark. The case $b = 0$, $n = 2k + 2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.

*Proof.* By [9], p. 313 the radial part of $L$ is

$$\begin{aligned}\Delta &= \partial_r^2 + \frac{A'(r)}{A(r)}\partial_r = A(r)^{-1}\circ\partial_r\circ A(r)\circ\partial_r \\ &= \partial_r^2 + ((n-1)\varepsilon\coth\varepsilon r + q\varepsilon\tanh\varepsilon r)\,\partial_r \\ &= \partial_r^2 + (p\varepsilon\coth\varepsilon r + 2q\varepsilon\coth 2\varepsilon r)\,\partial_r\,.\end{aligned}$$

The proof of the proposition breaks up into a few easy facts. First we have, for any $a, b \in \mathbf{R}$, the following equality of functions of $r > 0$:

$$\begin{aligned}(6)\quad &\big(\Delta - \varepsilon^2(a+b)(a+n+b+q-1)\big)f_{a,b} \\ &\qquad = a(a+n-2)f_{a-2,b} - \varepsilon^2 b(b+q-1)f_{a,b-2}\,,\end{aligned}$$

which is immediate from $\Delta f = A^{-1}(Af')'$ and the identities

$$f'_{a,b} = af_{a-1,b+1} + \varepsilon^2 bf_{a+1,b-1}\,,\quad f_{a,b} = f_{a,b-2} + \varepsilon^2 f_{a+2,b-2}\,.$$

LEMMA 7. *For $a+n \geq 2$, $\varepsilon = 0$ or $1$, the locally integrable function $f_{a,b}$ defines a $K$-invariant distribution $F_{a,b}$ on $X$ such that*

$$\begin{aligned}&\big(L - \varepsilon^2(a+b)(a+n+b+q-1)\big)\,F_{a,b} \\ &\quad = \begin{cases} a(a+n-2)F_{a-2,b} - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n > 2 \\ \omega_n a\,\delta - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n = 2\end{cases}\end{aligned}$$

*(equality of distributions on $X$).*

EXAMPLE. Taking $b = 0$, resp. $b = 1-q$, the lemma provides the following fundamental solutions (which coincide for $q = 1$)

$$\big(L + \varepsilon^2(n-2)(q+1)\big)\left(\frac{\sinh\varepsilon r}{\varepsilon}\right)^{2-n} = \omega_n(2-n)\delta$$

$$\big(L + 2\varepsilon^2(n+q-3)\big)\left(\frac{\sinh\varepsilon r}{\varepsilon}\right)^{2-n}(\cosh\varepsilon r)^{1-q} = \omega_n(2-n)\delta\,.$$

In the flat case $\varepsilon = 0$ they both reduce to $Lr^{2-n} = \omega_n(2-n)\delta$, a classical result for $\mathbf{R}^n$.

*Proof of Lemma 7.* Due to the $K$-invariance of $f_{a,b}$ and $L$ we need only consider $K$-invariant test functions $u \in \mathcal{D}(X)$. The integral

$$\int_X f_{a,b}\cdot u\,dx = \int_0^\infty f_{a,b}(r)u(r)A(r)\,dr = \omega_n\int_0^\infty f_{a+n-1,b+q}(r)u(r)\,dr\,,$$

absolutely convergent if $a+n>0$, defines a distribution $F_{a,b}$ on $X$. In view of the symmetry and $K$-invariance of the Laplace operator we have

$$\begin{aligned}\langle LF_{a,b}, u\rangle &= \langle F_{a,b}, Lu\rangle \\ &= \int_0^\infty f_{a,b}(r)\Delta u(r)A(r)\,dr = \int_0^\infty f_{a,b}(Au')'\,dr \\ &= \left(Af'_{a,b}u\right)(0) - \left(Af_{a,b}u'\right)(0) + \int_0^\infty (Af'_{a,b})'u\,dr\,.\end{aligned}$$

If $a+n>2$ the function $Af_{a,b}$ vanishes to order $a+n-1$ at the origin, and $Af'_{a,b}$ to order $a+n-2$. Since $u(r)$ is smooth (this notation stands for $u(\operatorname{Exp} rH)$ with $\|H\|=1$), it follows that

$$\langle LF_{a,b}, u\rangle = \int_0^\infty \Delta f_{a,b}(r)u(r)A(r)\,dr\,,$$

whence the result by (6).

The case $a+n=2$ is similar, in view of $(Af'_{a,b})(0)=\omega_n a$. □

Proposition 6 now follows easily: letting

$$L_a = L - \varepsilon^2(a+b)(a+n+b+q-1)$$

we have, by Lemma 7,

$$L_a F_{a,b} = \begin{cases} a(a+n-2)\,F_{a-2,b} & \text{if } a+n>2 \\ \omega_n a\,\delta & \text{if } a+n=2 \end{cases}$$

whenever $\varepsilon^2 b(b+q-1)=0$. Since

$$P_k(L) = L_{2-n}L_{4-n}\cdots L_{2k-n}\,,$$

the proposition follows by induction on $k$. □

THEOREM 8. *The $d$-geodesic Radon transform on a $n$-dimensional non-compact Riemannian isotropic space $X$ can be inverted by means of a polynomial of its Laplace-Beltrami operator $L$, under the following assumptions:*

(i) *$d$ is even: $d=2k$ with $k\geq 1$;*

(ii) *$X=\mathbf{R}^n$, or $\dim\mathfrak{s}_{2\alpha}=\dim\mathfrak{p}_{2\alpha}$, or else $\dim\mathfrak{s}_{2\alpha}=1$.*

*Then*

$$Cu = P_k(L)R^*Ru\,,$$

*for any $u\in\mathcal{D}(X)$, where $P_k$ is the polynomial from Proposition 6 (with $\varepsilon=1$, $q=\dim\mathfrak{p}_{2\alpha}$ and $b+q=\dim\mathfrak{s}_{2\alpha}$ if $X$ is hyperbolic, or $\varepsilon=0$ if $X=\mathbf{R}^n$) and*

$$C = \omega_d(-1)^k 2^{k-1}(k-1)!(n-2)(n-4)\cdots(n-2k)\,.$$

*Proof.* By (5) one has $S = (\omega_d/\omega_n) f_{a,b}$, with $a = d - n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (Section 4.1 **b**). The theorem follows from Proposition 6 and Section 3.2. $\square$

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9], Chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial $P_k$ is related to representation theory.

### 4.3 Examples

According to assumption (ii), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbf{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space $Y$ consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbf{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

**a**. A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8], p. 335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is $2$, $4$ or $8$; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

**b**. Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace $H$ by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbf{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}, \quad \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}.$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7], p. 171, or [9], p. 168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

**c**. Totally geodesic transform on classical hyperbolic spaces. Let $X = H^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1, m; \mathbf{F})$, $K = U(1; \mathbf{F}) \times U(m; \mathbf{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices

$$V = \begin{pmatrix} 0 & \overline{V}_1 & \cdots & \overline{V}_m \\ V_1 & & & \\ \vdots & & (0) & \\ V_m & & & \end{pmatrix}, \qquad V_i \in \mathbf{F},$$

can be identified with $\mathbf{F}^m$.

Let $\overline{V} \cdot W = \sum_{i=1}^m \overline{V}_i W_i$. For $U, V, W \in \mathfrak{p} = \mathbf{F}^m$, easy computations lead to

$$[U,[V,W]] = U\left(\overline{V} \cdot W - \overline{W} \cdot V\right) - V(\overline{W} \cdot U) + W(\overline{V} \cdot U) \tag{7}$$

($\mathbf{F}^m$ being considered as a $\mathbf{F}$-vector space, with scalars acting on the right). It follows that *any* $\mathbf{F}$*-subspace* $\mathfrak{s}$ *of* $\mathfrak{p}$ *is a Lie triple system.* Similarly, the natural inclusions $\mathbf{R}^m \subset \mathbf{C}^m \subset \mathbf{H}^m$ show that any $\mathbf{R}$-subspace of $\mathfrak{p} \cap \mathbf{R}^m$, or any $\mathbf{C}$-subspace of $\mathfrak{p} \cap \mathbf{C}^m$, is a Lie triple system.

Let $H \neq 0$ be an element of $\mathfrak{p}$. The eigenspaces of $(\operatorname{ad} H)^2$ can be obtained from (7), whence the decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}, \quad \mathfrak{a} = \mathbf{R}H,$$
$$\mathfrak{p}_\alpha = \{V \in \mathfrak{p} \mid \overline{H} \cdot V = 0\}, \quad \mathfrak{p}_{2\alpha} = \{H\lambda \mid \lambda \in \mathbf{F},\ \lambda + \overline{\lambda} = 0\},$$

with respective eigenvalues $0$, $\overline{H} \cdot H$ and $4\left(\overline{H} \cdot H\right)$. A similar decomposition holds for $\mathfrak{s}$, if $H$ is chosen in $\mathfrak{s}$. The spaces $\mathfrak{a} \oplus \mathfrak{p}_{2\alpha} = H\mathbf{F}$ and $\mathfrak{p}_\alpha$ are $\mathbf{F}$-subspaces of $\mathfrak{p}$, therefore Lie triple systems (as mentioned in a and b above). More generally, Theorem 8 applies to the following four families of totally geodesic submanifolds $\operatorname{Exp} \mathfrak{s}$; all superscripts in the table are real dimensions, with $k, l, m$ strictly positive integers.

| $X$ | $\dim \mathfrak{p}_\alpha$ | $\dim \mathfrak{p}_{2\alpha}$ | $\mathfrak{s}$ | $\dim \mathfrak{s}_\alpha$ | $\dim \mathfrak{s}_{2\alpha}$ | $y_o = \operatorname{Exp} \mathfrak{s}$ |
|---|---|---|---|---|---|---|
| $H^m(\mathbf{R})$ | $m-1$ | $0$ | (1) | $2k-1$ | $0$ | $H^{2k}(\mathbf{R})$ |
| $H^{2m}(\mathbf{C})$ | $2m-2$ | $1$ | (2) | $2k-2$ | $1$ | $H^{2k}(\mathbf{C})$ |
| $H^{4m}(\mathbf{H})$ | $4m-4$ | $3$ | (3) | $2k-2$ | $1$ | $H^{2k}(\mathbf{C})$ |
| $H^{4m}(\mathbf{H})$ | $4m-4$ | $3$ | (4) | $4l-4$ | $3$ | $H^{4l}(\mathbf{H})$ |

Case (1): $\mathfrak{s}$ is any $\mathbf{R}$-subspace of $\mathfrak{p} = \mathbf{R}^m$, with $\dim_{\mathbf{R}} \mathfrak{s} = 2k \leq m$.
Case (2): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathfrak{p} = \mathbf{C}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (3): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathbf{C}^m \subset \mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (4): $\mathfrak{s}$ is any $\mathbf{H}$-subspace of $\mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{H}} \mathfrak{s} = l \leq m$.

**d.** HOROCYCLE TRANSFORM ON REAL HYPERBOLIC SPACES. Proposition 6 also applies to this case, because of the similarity between the functions $S$ obtained in Propositions 4 and 5.

Following the same steps as for geodesic submanifolds, one can find a polynomial of the Laplacian with fundamental solution $S$ (case $q = 0$ in Proposition 5). Indeed $S(r)$ is now, up to a constant factor, $f_{-1,2-n}(r/2)$ in the notation of Section 4.2 with $\varepsilon = 1$. Let

$$\Delta_{p,q} = \partial_r^2 + (p \coth r + 2q \coth 2r)\, \partial_r$$

be the radial part of the Laplacian and $g(r) = f(r/2)$. Then

$$4\left(\Delta_{p,0}\, g\right)(r) = \left(\Delta_{0,p} f\right)\left(r/2\right) \ ;$$

note that the roles of $p$ and $q$ have been interchanged. The next theorem now follows from Propositions 5 and 6, with $n = 2k+1$, $\varepsilon = 1$ and $b = 1 - p = 2 - n$.

THEOREM 9 (Helgason). *The horocycle Radon transform on the odd-dimensional hyperbolic space $X = H^{2k+1}(\mathbf{R})$, $k \geq 1$, is inverted by*

$$Cu = Q_k(L) R^* R u \,,$$

*where $u \in \mathcal{D}(X)$, $L$ is the Laplace-Beltrami operator of $X$,*

$$C = \left(-\frac{\pi}{2}\right)^k \frac{(2k-1)!}{(k-1)!}\,, \quad Q_k(x) = \prod_{j=1}^{k} (x + j(2k-j)) \ .$$

In [11], p. 210, the normalization of the Riemannian metric on $X$ differs from ours.

The result extends to the horocycle transform on a Riemannian symmetric space $X = G/K$ of the noncompact type, provided that the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below). The spaces $H^{2k+1}(\mathbf{R})$ in Theorem 9 are the rank one spaces among those.