Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

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DOI: https://doi.org/10.5169/seals-65436

$$\int_N u(n\cdot x_o)\,dn = 2^{p-2+(q/2)}\omega_p\omega_q \int_0^\infty\int_0^\infty u(\exp(rH)\cdot x_o)\,x^{(p/2)-1}y^{(q/2)-1}dx\,dy$$
$$= \int_0^\infty u(\exp(rH)\cdot x_o) f(r)\,dr\,.$$

The latter expression follows from the change of variables $(x,r)\mapsto(x,y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)}\omega_p\omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1}\left(\cosh^2 r-(1+x)^2\right)^{(q/2)-1}dx\,.$$

Setting $x = t(\cosh r - 1)$ we find

$$f(r) = 2^{(3p+q)/2}\omega_{n-1}\left(\sinh r\right)^{q-1}\left(\sinh\frac{r}{2}\right)^p \cosh r$$
$$\times \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\int_0^1 t^{(p/2)-1}(1-t)^{(q/2)-1}\left(1+t\tanh^2\frac{r}{2}\right)^{(q/2)-1}dt$$
$$= 2^{(3p+q)/2}\omega_{n-1}\left(\sinh r\right)^{q-1}\left(\sinh\frac{r}{2}\right)^p \cosh r$$
$$\times\, {}_2F_1\left(\frac{p}{2},1-\frac{q}{2};\frac{p+q}{2};-\tanh^2\frac{r}{2}\right),$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2}\omega_{n-1}(\sinh r)^{n-2}(\cosh r)^q\,{}_2F_1\left(\frac{\rho-1}{2},\frac{\rho}{2};\frac{n-1}{2};-\sinh^2 r\right).$$

Thus, for K-invariant u,

$$\int_N u(n\cdot x_o)\,dn = \int_0^\infty u\left(\exp(rH)\cdot x_o\right)\,S(r)A(r)\,dr = \int_X u(x)\,S(x)\,dx\,,$$

where $A(r) = \omega_n(\sinh r)^{n-1}(\cosh r)^q$ and $S(r) = f(r)/A(r)$. □

3.2 Radon inversion by convolution

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u * S = R^*Ru$, in the form

$$u = DR^*Ru\,. \tag{2}$$

To recover $u(x)$ from Ru the recipe will be to integrate $Ru(y)$ over all y incident to x, and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, R^*R commutes with the action of G on X, and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S*T=\delta$, then

$$u = (R^* Ru) * T \,.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G-invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \big\langle u(g \cdot x_o), \langle S, ({}^t D\varphi) \circ \tau(g) \rangle \big\rangle \qquad \text{by (1)} \\ &= \big\langle u(g \cdot x_o), \langle S, {}^t D(\varphi \circ \tau(g)) \rangle \big\rangle \,, \end{aligned}$$

since the transpose operator ${}^t D$ is G-invariant too, as follows from the existence of a G-invariant measure on X. Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \big\langle u(g \cdot x_o), \langle DS, \varphi \circ \tau(g) \rangle \big\rangle \\ &= \langle u * DS, \varphi \rangle \,, \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can *find a G-invariant differential operator D on X with given fundamental solution S*. We shall now investigate this question on the basis of Propositions 4 and 5.

4. Radon transforms on isotropic spaces

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}),\ H^{2m}(\mathbf{C}),\ H^{4m}(\mathbf{H}),\ H^{16}(\mathbf{O}) \,,$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], § 8.12). We first try to invert the d-geodesic Radon transform on X, defined by integrating over a family of d-dimensional totally geodesic submanifolds of X. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 Totally geodesic submanifolds

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.