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Autor: Rouvière, François
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$$\begin{aligned} \int_N u(n \cdot x_o) dn &= 2^{p-2+(q/2)} \omega_p \omega_q \int_0^\infty \int_0^\infty u(\exp(rH) \cdot x_o) x^{(p/2)-1} y^{(q/2)-1} dx dy \\ &= \int_0^\infty u(\exp(rH) \cdot x_o) f(r) dr. \end{aligned}$$

The latter expression follows from the change of variables $(x, r) \mapsto (x, y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)} \omega_p \omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1} (\cosh^2 r - (1+x)^2)^{(q/2)-1} dx.$$

Setting $x = t(\cosh r - 1)$ we find

$$\begin{aligned} f(r) &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \int_0^1 t^{(p/2)-1} (1-t)^{(q/2)-1} \left(1 + t \tanh^2 \frac{r}{2}\right)^{(q/2)-1} dt \\ &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times {}_2F_1\left(\frac{p}{2}, 1 - \frac{q}{2}; \frac{p+q}{2}; -\tanh^2 \frac{r}{2}\right), \end{aligned}$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2} \omega_{n-1} (\sinh r)^{n-2} (\cosh r)^q {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right).$$

Thus, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\exp(rH) \cdot x_o) S(r) A(r) dr = \int_X u(x) S(x) dx,$$

where $A(r) = \omega_n (\sinh r)^{n-1} (\cosh r)^q$ and $S(r) = f(r)/A(r)$. \square

3.2 RADON INVERSION BY CONVOLUTION

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u * S = R^* R u$, in the form

$$(2) \quad u = D R^* R u.$$

To recover $u(x)$ from $R u$ the recipe will be to integrate $R u(y)$ over all y incident to x , and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, $R^* R$ commutes with the action of G on X , and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S * T = \delta$, then

$$u = (R^*Ru) * T.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G -invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ ,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^t D\varphi \rangle \\ &= \langle u(g \cdot x_0), \langle S, ({}^t D\varphi) \circ \tau(g) \rangle \rangle \quad \text{by (1)} \\ &= \langle u(g \cdot x_0), \langle S, {}^t D(\varphi \circ \tau(g)) \rangle \rangle, \end{aligned}$$

since the transpose operator ${}^t D$ is G -invariant too, as follows from the existence of a G -invariant measure on X . Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u(g \cdot x_0), \langle DS, \varphi \circ \tau(g) \rangle \rangle \\ &= \langle u * DS, \varphi \rangle, \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can find a G -invariant differential operator D on X with given fundamental solution S . We shall now investigate this question on the basis of Propositions 4 and 5.

4. RADON TRANSFORMS ON ISOTROPIC SPACES

Throughout this section X will be an isotropic connected noncompact Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}), H^{2m}(\mathbf{C}), H^{4m}(\mathbf{H}), H^{16}(\mathbf{O}),$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the d -geodesic Radon transform on X , defined by integrating over a family of d -dimensional totally geodesic submanifolds of X . At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 TOTALLY GEODESIC SUBMANIFOLDS

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.