

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH
Autor: Rouvière, François
Kapitel: 3.1 A CONVOLUTION FORMULA
DOI: <https://doi.org/10.5169/seals-65436>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. CONVOLUTION ON X AND INVERSION OF R

3.1 A CONVOLUTION FORMULA

Again G is a Lie group, K a compact subgroup, $X = G/K$ and $\tau(g)$ denotes the natural action of G on X , i.e. $\tau(g)x = g \cdot x$.

a. A GENERAL RESULT. Let $S_1, S_2 \in \mathcal{D}'(X)$ be two distributions on X , with S_2 assumed K -invariant. By analogy with the group case (if K were the trivial subgroup), the convolution $S_1 * S_2 \in \mathcal{D}'(X)$ can be defined by

$$(1) \quad \begin{aligned} \langle S_1 * S_2, \varphi \rangle &= \langle S_1(g_1K), \langle S_2(g_2K), \varphi(g_1g_2K) \rangle \rangle \\ &= \langle S_1(g_1K), \langle S_2, \varphi \circ \tau(g_1) \rangle \rangle, \end{aligned}$$

for any $\varphi \in \mathcal{D}(X)$. Indeed, the K -invariance of S_2 implies that $\langle S_2, \varphi \circ \tau(g_1) \rangle$ is a right K -invariant function of $g_1 \in G$, hence defines a function of $g_1K \in X$ to which S_1 can be applied (assuming that S_1 or S_2 has compact support). A more classical definition ([9], p. 290) of $S_1 * S_2$ arises from the convolution on the group G itself, by means of the projection $G \rightarrow G/K$; it is easily checked that both definitions agree, but (1) will be more convenient here (and could be used even if K were not compact).

PROPOSITION 3. Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G -invariant measure. For any $u \in C_c(X)$ we have

$$R^*Ru = u * S,$$

a convolution on X . Here, denoting by δ the Dirac measure at the origin $x_o = K$ of X , the distribution $S = R^*R\delta$ is the K -invariant measure on X given by

$$\langle S, u \rangle = R^*Ru(x_o) = \int_{K \times H} u(kh \cdot x_o) dk dh = Ru_K(y_o),$$

with $u_K(x) = \int_K u(k \cdot x) dk$ and $y_o = H$.

Proof. The definition of the Radon transforms R and R^* clearly show that they intertwine the actions of G on X and Y (here denoted by $\tau_X(g)$, resp. $\tau_Y(g)$, for $g \in G$):

$$R(u \circ \tau_X(g)) = (Ru) \circ \tau_Y(g), \quad R^*(v \circ \tau_Y(g)) = (R^*v) \circ \tau_X(g).$$

Therefore R^*R commutes with $\tau_X(g)$, hence is a right convolution operator. Indeed, let $\varphi \in \mathcal{D}(X)$ be a test function. The distribution S defined by

$\langle S, \varphi \rangle = R^*R\varphi(x_0)$ extends to a K -invariant positive linear form on $C_c(X)$, i.e. a measure, and

$$\begin{aligned} \langle u * S, \varphi \rangle &= \langle u(g \cdot x_0), \langle S, \varphi \circ \tau_X(g) \rangle \rangle && \text{by (1)} \\ &= \langle u(g \cdot x_0), R^*R(\varphi \circ \tau_X(g))(x_0) \rangle \\ &= \langle u(g \cdot x_0), (R^*R\varphi)(g \cdot x_0) \rangle \\ &= \langle u, R^*R\varphi \rangle = \langle R^*Ru, \varphi \rangle. \end{aligned}$$

The last equality follows from the duality between R and R^* (Proposition 2). \square

b. TOTALLY GEODESIC TRANSFORM ON ISOTROPIC SPACES. The following variant of Proposition 3 gives a more precise statement in a specific situation. Unifying and extending several results from the literature on totally geodesic Radon transforms on two-point homogeneous spaces (Helgason [9], p.104, 124 and 160, Berenstein and Casadio Tarabusi [1] p.618), it will lead to inversion formulas. Let $X = G/K$ be an *isotropic* connected non compact Riemannian manifold with distance d , where G is a transitive Lie group of isometries of X and K is the isotropy subgroup of some origin $x_0 \in X$. Let y_0 be a *totally geodesic* submanifold of X , containing x_0 , and let Y be the set of all submanifolds $y = g \cdot y_0$ of X , with $g \in G$. We denote by $A(r)$, resp. $A_o(r)$, the Riemannian measure (area) of a sphere of radius r in X , resp. in y_0 .

As explained in Section 4.1 a below, Lemma 1 applies to this situation and the Radon transform can be written as

$$Ru(y) = \int_y u(x) dm_y(x), \quad u \in C_c(X), \quad y \in Y,$$

where dm_y is the Riemannian measure induced by X on its submanifold y , and

$$R^*v(g \cdot x_0) = \int_K v(gk \cdot y_0) dk, \quad v \in C(Y), \quad g \in G.$$

Note that we will not need here the group H nor an invariant measure on G/H , as opposed to Proposition 3.

PROPOSITION 4. *With the above notation we have, for any $u \in C_c(X)$,*

$$R^*Ru = u * S$$

(convolution on X), where S is the K -invariant function on X defined by

$$S(x) = A_o(r)/A(r), \quad r = d(x_0, x).$$

An explicit formula (4) for S will be given in Section 4.1, after we introduce the relevant notations.

Proof. Fix $z = g \cdot x_o \in X$. The measure dm_y on $y = gk \cdot y_o$ corresponds to the measure dm_o on y_o by the isometry $x \mapsto gk \cdot x$, whence

$$R^*Ru(z) = \int_{y_o} \left(\int_K u(gk \cdot x) dk \right) dm_o(x).$$

Now, X being isotropic, K -orbits are spheres centered at x_o . Since $\int_K dk = 1$, the above integral over K is the mean value $(M_ru)(z)$ of u over the sphere $\Sigma(z, r)$ with center z and radius $r = d(x_o, x)$. Therefore

$$\int_K u(gk \cdot x) dk = (M_ru)(z) = \frac{1}{A(r)} \int_{\Sigma(z,r)} u d\sigma,$$

where $d\sigma$ is the Riemannian measure on $\Sigma(z, r)$, and

$$R^*Ru(z) = \int_{y_o} (M_ru)(z) dm_o(x).$$

But, y_o being totally geodesic, the distance $r = d(x_o, x)$ between two points of y_o is the same in X and in y_o , and the latter integral can thus be computed in geodesic polar coordinates on y_o (with center x_o), as

$$\begin{aligned} R^*Ru(z) &= \int_0^\infty (M_ru)(z) A_o(r) dr \\ &= \int_0^\infty (M_ru)(z) A(r) f(r) dr \end{aligned}$$

with $f(r) = A_o(r)/A(r)$. This in turn can be viewed as an integral over X computed in polar coordinates (with center z), namely

$$R^*Ru(z) = \int_0^\infty f(r) dr \int_{\Sigma(z,r)} u d\sigma = \int_X u(x) f(d(z, x)) dx.$$

Setting $z = g \cdot x_o$, $x = g' \cdot x_o$ it follows that, for any test function $\varphi \in \mathcal{D}(X)$,

$$\int_X R^*Ru(z) \varphi(z) dz = \int_{G \times G} u(g' \cdot x_o) f(d(g \cdot x_o, g' \cdot x_o)) \varphi(g \cdot x_o) dg' dg.$$

Changing the variable g into $g = g'g''$ (with fixed g') in $\int dg$, we obtain from the left invariance of dg

$$\begin{aligned} \int_X R^*Ru(z) \varphi(z) dz &= \int_{G \times G} u(g' \cdot x_o) f(d(g'' \cdot x_o, x_o)) \varphi(g'g'' \cdot x_o) dg' dg'' \\ &= \langle u * S, \varphi \rangle, \end{aligned}$$

according to (1) and the definition of S in the proposition. □

c. HOROCYCLE TRANSFORM ON RANK ONE SPACES. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type, $G = KAN$ an Iwasawa decomposition (cf. Notations, d) and $Y = G/MN$ the space of all horocycles in X . The corresponding dual Radon transforms are

$$Ru(gMN) = \int_N u(gnK) dn, \quad R^*v(gK) = \int_K v(gkN) dk$$

for $u \in C_c(X)$, $v \in C(Y)$; MN has been replaced by N in the right-hand sides because K contains M .

We now specialize to rank one spaces, with positive roots α and (possibly) 2α . Let H be the basis vector of \mathfrak{a} such that $\alpha(H) = 1$. Multiplying the Killing form scalar product on \mathfrak{g} by a suitable factor, it will be convenient to assume that the corresponding norm on \mathfrak{p} satisfies $\|H\| = 1$.

The exponential mapping $\exp : \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \rightarrow N$ is a diffeomorphism onto, with Jacobian 1; the Haar measure dn on N can therefore be chosen so that

$$\int_N f(n) dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} f(\exp(Z + T)) dZdT,$$

where dZ , resp. dT , is the Lebesgue measure on \mathfrak{g}_α , resp. $\mathfrak{g}_{2\alpha}$, corresponding to the norm $\|\cdot\|$.

Let $p = \dim \mathfrak{g}_\alpha$, $q = \dim \mathfrak{g}_{2\alpha}$, $\rho = (p/2) + q$, $n = p + q + 1 = \dim X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. With the above normalizations we now have the following analogue of Proposition 4.

PROPOSITION 5. *For the horocycle Radon transform on X , a rank one Riemannian symmetric space of the noncompact type, and $u \in C_c(X)$ we have*

$$R^*Ru = u * S,$$

(convolution on X). Here S is the radial function on X given by

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right),$$

with $r > 0$. For $X = H^n(\mathbf{R})$, i.e. $q = 0$, this reduces to

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} \left(\cosh \frac{r}{2}\right)^{3-n}.$$

Proof. We first assume $q = 0$. The groups G and MN being unimodular, the space $Y = G/MN$ has a G -invariant measure ([11], p.100). By Proposition 3 it follows that $R^*Ru = u * S$, with

$$\langle S, u \rangle = \int_N u(n \cdot x_o) dn = \int_{\mathfrak{g}_\alpha} u(\exp Z \cdot x_o) dZ$$

for any K -invariant function u on X (this will suffice to find the K -invariant function S).

By classical rank one computations ([8], p.414), the radial component $\exp(rH)$ of $\exp Z$ is given by

$$\exp Z \cdot x_o = k \exp(rH) \cdot x_o,$$

with $k \in K$, $r \geq 0$ and $\|Z\| = 2\sqrt{2} \sinh(r/2)$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^{n-1}$ it follows that, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\text{Exp } rH) f(r) dr,$$

with

$$f(r) = 2^{(3/2)(n-1)-1} \omega_{n-1} \left(\sinh \frac{r}{2} \right)^{n-2} \cosh \frac{r}{2}.$$

On the other hand, using the diffeomorphism Exp and spherical coordinates on \mathfrak{p} we have

$$\int_X u(x) dx = \int_0^\infty u(\text{Exp } rH) A(r) dr, \text{ with } A(r) = \omega_n (\sinh r)^{n-1}$$

(cf. Section 4.1 **b** for more details). If $S(r) = f(r)/A(r)$ we thus have, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\text{Exp } rH) S(r) A(r) dr = \int_X u(x) S(x) dx,$$

as claimed.

The case $q \geq 1$ will not be used in the sequel; we sketch its proof, similar to that of the case $q = 0$. First

$$\langle S, u \rangle = \int_N u(n \cdot x_o) dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} u(\exp(Z + T) \cdot x_o) dZ dT.$$

Then, by rank one computations ([8], p.414),

$$\begin{aligned} \exp(Z + T) \cdot x_o &= k \exp(rH) \cdot x_o, \quad k \in K, \\ \cosh^2 r &= \left(1 + \frac{1}{4} \|Z\|^2 \right)^2 + \frac{1}{2} \|T\|^2, \quad r \geq 0. \end{aligned}$$

Let $x = \|Z\|^2/4$, $y = \|T\|^2/2$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^p$ and $\mathfrak{g}_{2\alpha} = \mathbf{R}^q$ we obtain

$$\begin{aligned} \int_N u(n \cdot x_o) dn &= 2^{p-2+(q/2)} \omega_p \omega_q \int_0^\infty \int_0^\infty u(\exp(rH) \cdot x_o) x^{(p/2)-1} y^{(q/2)-1} dx dy \\ &= \int_0^\infty u(\exp(rH) \cdot x_o) f(r) dr. \end{aligned}$$

The latter expression follows from the change of variables $(x, r) \mapsto (x, y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)} \omega_p \omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1} (\cosh^2 r - (1+x)^2)^{(q/2)-1} dx.$$

Setting $x = t(\cosh r - 1)$ we find

$$\begin{aligned} f(r) &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \int_0^1 t^{(p/2)-1} (1-t)^{(q/2)-1} \left(1 + t \tanh^2 \frac{r}{2}\right)^{(q/2)-1} dt \\ &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times {}_2F_1\left(\frac{p}{2}, 1 - \frac{q}{2}; \frac{p+q}{2}; -\tanh^2 \frac{r}{2}\right), \end{aligned}$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2} \omega_{n-1} (\sinh r)^{n-2} (\cosh r)^q {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right).$$

Thus, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\exp(rH) \cdot x_o) S(r) A(r) dr = \int_X u(x) S(x) dx,$$

where $A(r) = \omega_n (\sinh r)^{n-1} (\cosh r)^q$ and $S(r) = f(r)/A(r)$. \square

3.2 RADON INVERSION BY CONVOLUTION

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u * S = R^* R u$, in the form

$$(2) \quad u = D R^* R u.$$

To recover $u(x)$ from Ru the recipe will be to integrate $Ru(y)$ over all y incident to x , and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, $R^* R$ commutes with the action of G on X , and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S * T = \delta$, then