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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

Autor: Rouvière, François

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3. Convolution on X and inversion of R

3.1 A convolution formula

Again G is a Lie group, K a *compact* subgroup, $X = G/K$ and $\tau(g)$ denotes the natural action of G on X, i.e. $\tau(g)x = g \cdot x$.

a. A GENERAL RESULT. Let $S_1, S_2 \in \mathcal{D}'(X)$ be two distributions on X, with S_2 assumed K-invariant. By analogy with the group case (if K were the trivial subgroup), the convolution $S_1 * S_2 \in \mathcal{D}'(X)$ can be defined by

$$\begin{aligned}(1)\qquad \langle S_1 * S_2, \varphi\rangle &= \big\langle S_1(g_1K), \langle S_2(g_2K), \varphi(g_1g_2K)\rangle\big\rangle \\ &= \big\langle S_1(g_1K), \langle S_2, \varphi \circ \tau(g_1)\rangle\big\rangle\,,\end{aligned}$$

for any $\varphi \in \mathcal{D}(X)$. Indeed, the K-invariance of S_2 implies that $\langle S_2, \varphi \circ \tau(g_1)\rangle$ is a right K-invariant function of $g_1 \in G$, hence defines a function of $g_1K \in X$ to which S_1 can be applied (assuming that S_1 or S_2 has compact support). A more classical definition ([9], p. 290) of $S_1 * S_2$ arises from the convolution on the group G itself, by means of the projection $G \to G/K$; it is easily checked that both definitions agree, but (1) will be more convenient here (and could be used even if K were not compact).

PROPOSITION 3. *Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G-invariant measure. For any $u \in C_c(X)$ we have*

$$R^*Ru = u * S\,,$$

a convolution on X. Here, denoting by δ the Dirac measure at the origin $x_o = K$ of X, the distribution $S = R^*R\delta$ is the K-invariant measure on X given by

$$\langle S, u\rangle = R^*Ru(x_o) = \int_{K\times H} u(kh \cdot x_o)\, dk\, dh = Ru_K(y_o)\,,$$

with $u_K(x) = \int_K u(k \cdot x)\, dk$ and $y_o = H$.

Proof. The definition of the Radon transforms R and R^* clearly show that they intertwine the actions of G on X and Y (here denoted by $\tau_X(g)$, resp. $\tau_Y(g)$, for $g \in G$):

$$R(u \circ \tau_X(g)) = (Ru) \circ \tau_Y(g)\,, \quad R^*(v \circ \tau_Y(g)) = (R^*v) \circ \tau_X(g)\,.$$

Therefore R^*R commutes with $\tau_X(g)$, hence is a right convolution operator. Indeed, let $\varphi \in \mathcal{D}(X)$ be a test function. The distribution S defined by

$\langle S, \varphi \rangle = R^*R\varphi(x_o)$ extends to a K-invariant positive linear form on $C_c(X)$, i.e. a measure, and

$$\begin{aligned} \langle u * S, \varphi \rangle &= \langle u(g \cdot x_o), \langle S, \varphi \circ \tau_X(g) \rangle \rangle \qquad \text{by (1)} \\ &= \langle u(g \cdot x_o), R^*R(\varphi \circ \tau_X(g))(x_o) \rangle \\ &= \langle u(g \cdot x_o), (R^*R\varphi)(g \cdot x_o) \rangle \\ &= \langle u, R^*R\varphi \rangle = \langle R^*Ru, \varphi \rangle . \end{aligned}$$

The last equality follows from the duality between R and R^* (Proposition 2). □

b. TOTALLY GEODESIC TRANSFORM ON ISOTROPIC SPACES. The following variant of Proposition 3 gives a more precise statement in a specific situation. Unifying and extending several results from the literature on totally geodesic Radon transforms on two-point homogeneous spaces (Helgason [9], p. 104, 124 and 160, Berenstein and Casadio Tarabusi [1] p. 618), it will lead to inversion formulas. Let $X = G/K$ be an *isotropic* connected non compact Riemannian manifold with distance d, where G is a transitive Lie group of isometries of X and K is the isotropy subgroup of some origin $x_o \in X$. Let y_o be a *totally geodesic* submanifold of X, containing x_o, and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$. We denote by $A(r)$, resp. $A_o(r)$, the Riemannian measure (area) of a sphere of radius r in X, resp. in y_o.

As explained in Section 4.1 **a** below, Lemma 1 applies to this situation and the Radon transform can be written as

$$Ru(y) = \int_y u(x)\, dm_y(x)\,, \quad u \in C_c(X)\,, \quad y \in Y\,,$$

where dm_y is the Riemannian measure induced by X on its submanifold y, and

$$R^*v(g \cdot x_o) = \int_K v(gk \cdot y_o)\, dk\,, \quad v \in C(Y),\ g \in G\,.$$

Note that we will not need here the group H nor an invariant measure on G/H, as opposed to Proposition 3.

PROPOSITION 4. *With the above notation we have, for any $u \in C_c(X)$,*

$$R^*Ru = u * S$$

(convolution on X), where S is the K-invariant function on X defined by

$$S(x) = A_o(r)/A(r)\,, \quad r = d(x_o, x)\,.$$

An explicit formula (4) for S will be given in Section 4.1, after we introduce the relevant notations.

Proof. Fix $z = g \cdot x_o \in X$. The measure dm_y on $y = gk \cdot y_o$ corresponds to the measure dm_o on y_o by the isometry $x \mapsto gk \cdot x$, whence

$$R^* Ru(z) = \int_{y_o} \left(\int_K u(gk \cdot x)\, dk \right) dm_o(x)\,.$$

Now, X being isotropic, K-orbits are spheres centered at x_o. Since $\int_K dk = 1$, the above integral over K is the mean value $(M_r u)(z)$ of u over the sphere $\Sigma(z, r)$ with center z and radius $r = d(x_o, x)$. Therefore

$$\int_K u(gk \cdot x)\, dk = (M_r u)(z) = \frac{1}{A(r)} \int_{\Sigma(z,r)} u\, d\sigma\,,$$

where $d\sigma$ is the Riemannian measure on $\Sigma(z, r)$, and

$$R^* Ru(z) = \int_{y_o} (M_r u)(z)\, dm_o(x)\,.$$

But, y_o being totally geodesic, the distance $r = d(x_o, x)$ between two points of y_o is the same in X and in y_o, and the latter integral can thus be computed in geodesic polar coordinates on y_o (with center x_o), as

$$\begin{aligned} R^* Ru(z) &= \int_0^\infty (M_r u)(z) A_o(r)\, dr \\ &= \int_0^\infty (M_r u)(z) A(r) f(r)\, dr \end{aligned}$$

with $f(r) = A_o(r)/A(r)$. This in turn can be viewed as an integral over X computed in polar coordinates (with center z), namely

$$R^* Ru(z) = \int_0^\infty f(r)\, dr \int_{\Sigma(z,r)} u\, d\sigma = \int_X u(x) f(d(z,x))\, dx\,.$$

Setting $z = g \cdot x_o$, $x = g' \cdot x_o$ it follows that, for any test function $\varphi \in \mathcal{D}(X)$,

$$\int_X R^* Ru(z)\, \varphi(z)\, dz = \int_{G \times G} u(g' \cdot x_o) f(d(g \cdot x_o, g' \cdot x_o))\, \varphi(g \cdot x_o)\, dg' dg\,.$$

Changing the variable g into $g = g'g''$ (with fixed g') in $\int dg$, we obtain from the left invariance of dg

$$\begin{aligned} \int_X R^* Ru(z)\, \varphi(z)\, dz &= \int_{G \times G} u(g' \cdot x_o) f(d(g'' \cdot x_o, x_o))\, \varphi(g'g'' \cdot x_o)\, dg' dg'' \\ &= \langle u * S, \varphi \rangle\,, \end{aligned}$$

according to (1) and the definition of S in the proposition. □

c. HOROCYCLE TRANSFORM ON RANK ONE SPACES. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type, $G = KAN$ an Iwasawa decomposition (cf. Notations, **d**) and $Y = G/MN$ the space of all horocycles in X. The corresponding dual Radon transforms are

$$Ru(gMN) = \int_N u(gnK)\, dn\,, \quad R^*v(gK) = \int_K v(gkN)\, dk$$

for $u \in C_c(X)$, $v \in C(Y)$; MN has been replaced by N in the right-hand sides because K contains M.

We now specialize to rank one spaces, with positive roots α and (possibly) 2α. Let H be the basis vector of $\mathfrak{a}$ such that $\alpha(H) = 1$. Multiplying the Killing form scalar product on $\mathfrak{g}$ by a suitable factor, it will be convenient to assume that the corresponding norm on $\mathfrak{p}$ satisfies $\|H\| = 1$.

The exponential mapping $\exp : \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to N$ is a diffeomorphism onto, with Jacobian 1; the Haar measure dn on N can therefore be chosen so that

$$\int_N f(n)\, dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} f(\exp(Z+T))\, dZdT\,,$$

where dZ, resp. dT, is the Lebesgue measure on $\mathfrak{g}_\alpha$, resp. $\mathfrak{g}_{2\alpha}$, corresponding to the norm $\| \ \|$.

Let $p = \dim \mathfrak{g}_\alpha$, $q = \dim \mathfrak{g}_{2\alpha}$, $\rho = (p/2) + q$, $n = p + q + 1 = \dim X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. With the above normalizations we now have the following analogue of Proposition 4.

PROPOSITION 5. *For the horocycle Radon transform on X, a rank one Riemannian symmetric space of the noncompact type, and $u \in C_c(X)$ we have*

$$R^*Ru = u * S\,,$$

(convolution on X). Here S is the radial function on X given by

$$S(r) = 2^{(n-1)/2}\frac{\omega_{n-1}}{\omega_n}(\sinh r)^{-1}\, {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right),$$

with $r > 0$. For $X = H^n(\mathbf{R})$, i.e. $q = 0$, this reduces to

$$S(r) = 2^{(n-1)/2}\frac{\omega_{n-1}}{\omega_n}(\sinh r)^{-1}\left(\cosh\frac{r}{2}\right)^{3-n}.$$

Proof. We first assume $q = 0$. The groups G and MN being unimodular, the space $Y = G/MN$ has a G-invariant measure ([11], p. 100). By Proposition 3 it follows that $R^*Ru = u * S$, with

$$\langle S, u\rangle = \int_N u(n\cdot x_o)\,dn = \int_{\mathfrak{g}_\alpha} u(\exp Z\cdot x_o)\,dZ$$

for any K-invariant function u on X (this will suffice to find the K-invariant function S).

By classical rank one computations ([8], p. 414), the radial component $\exp(rH)$ of $\exp Z$ is given by

$$\exp Z\cdot x_o = k\exp(rH)\cdot x_o\,,$$

with $k\in K$, $r\geq 0$ and $\|Z\| = 2\sqrt{2}\sinh(r/2)$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^{n-1}$ it follows that, for K-invariant u,

$$\int_N u(n\cdot x_o)\,dn = \int_0^\infty u(\mathrm{Exp}\ rH)f(r)\,dr\,,$$

with

$$f(r) = 2^{(3/2)(n-1)-1}\omega_{n-1}\left(\sinh\frac{r}{2}\right)^{n-2}\cosh\frac{r}{2}\,.$$

On the other hand, using the diffeomorphism Exp and spherical coordinates on $\mathfrak{p}$ we have

$$\int_X u(x)\,dx = \int_0^\infty u(\mathrm{Exp}\ rH)A(r)\,dr\ ,\ \text{with}\ A(r) = \omega_n(\sinh r)^{n-1}$$

(cf. Section 4.1 **b** for more details). If $S(r) = f(r)/A(r)$ we thus have, for K-invariant u,

$$\int_N u(n\cdot x_o)\,dn = \int_0^\infty u(\mathrm{Exp}\ rH)\,S(r)A(r)\,dr = \int_X u(x)\,S(x)\,dx\,,$$

as claimed.

The case $q\geq 1$ will not be used in the sequel; we sketch its proof, similar to that of the case $q = 0$. First

$$\langle S, u\rangle = \int_N u(n\cdot x_o)\,dn = \int_{\mathfrak{g}_\alpha\times\mathfrak{g}_{2\alpha}} u(\exp(Z+T)\cdot x_o)\,dZ\,dT\,.$$

Then, by rank one computations ([8], p. 414),

$$\exp(Z+T)\cdot x_o = k\exp(rH)\cdot x_o\,,\quad k\in K\,,$$

$$\cosh^2 r = \left(1+\frac{1}{4}\|Z\|^2\right)^2 + \frac{1}{2}\|T\|^2,\quad r\geq 0\,.$$

Let $x = \|Z\|^2/4$, $y = \|T\|^2/2$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^p$ and $\mathfrak{g}_{2\alpha} = \mathbf{R}^q$ we obtain

$$\int_N u(n\cdot x_o)\,dn = 2^{p-2+(q/2)}\omega_p\omega_q \int_0^\infty\int_0^\infty u(\exp(rH)\cdot x_o)\,x^{(p/2)-1}y^{(q/2)-1}dx\,dy$$
$$= \int_0^\infty u(\exp(rH)\cdot x_o)f(r)\,dr\,.$$

The latter expression follows from the change of variables $(x,r)\mapsto(x,y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)}\omega_p\omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1}\left(\cosh^2 r-(1+x)^2\right)^{(q/2)-1}dx\,.$$

Setting $x = t(\cosh r - 1)$ we find

$$f(r) = 2^{(3p+q)/2}\omega_{n-1}\left(\sinh r\right)^{q-1}\left(\sinh\frac{r}{2}\right)^p\cosh r$$
$$\times\frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\int_0^1 t^{(p/2)-1}(1-t)^{(q/2)-1}\left(1+t\tanh^2\frac{r}{2}\right)^{(q/2)-1}dt$$
$$= 2^{(3p+q)/2}\omega_{n-1}\left(\sinh r\right)^{q-1}\left(\sinh\frac{r}{2}\right)^p\cosh r$$
$$\times\,{}_2F_1\left(\frac{p}{2},1-\frac{q}{2};\frac{p+q}{2};-\tanh^2\frac{r}{2}\right),$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2}\omega_{n-1}(\sinh r)^{n-2}(\cosh r)^q\,{}_2F_1\left(\frac{\rho-1}{2},\frac{\rho}{2};\frac{n-1}{2};-\sinh^2 r\right).$$

Thus, for K-invariant u,

$$\int_N u(n\cdot x_o)\,dn = \int_0^\infty u\left(\exp(rH)\cdot x_o\right)\,S(r)A(r)\,dr = \int_X u(x)\,S(x)\,dx\,,$$

where $A(r) = \omega_n(\sinh r)^{n-1}(\cosh r)^q$ and $S(r) = f(r)/A(r)$. □

3.2 Radon inversion by convolution

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u*S = R^*Ru$, in the form

$$u = DR^*Ru\,. \tag{2}$$

To recover $u(x)$ from Ru the recipe will be to integrate $Ru(y)$ over all y incident to x, and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, R^*R commutes with the action of G on X, and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S*T=\delta$, then