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3. CONVOLUTION ON X AND INVERSION OF R

3.1 A CONVOLUTION FORMULA

Again G is a Lie group, K a compact subgroup, X = G/K and $\tau(g)$ denotes the natural action of G on X, i.e. $\tau(g)x = g \cdot x$.

a. A GENERAL RESULT. Let $S_1, S_2 \in \mathcal{D}'(X)$ be two distributions on X, with S_2 assumed K-invariant. By analogy with the group case (if K were the trivial subgroup), the convolution $S_1 * S_2 \in \mathcal{D}'(X)$ can be defined by

(1)
$$\langle S_1 * S_2, \varphi \rangle = \langle S_1(g_1K), \langle S_2(g_2K), \varphi(g_1g_2K) \rangle \rangle$$

= $\langle S_1(g_1K), \langle S_2, \varphi \circ \tau(g_1) \rangle \rangle,$

for any $\varphi \in \mathcal{D}(X)$. Indeed, the *K*-invariance of S_2 implies that $\langle S_2, \varphi \circ \tau(g_1) \rangle$ is a right *K*-invariant function of $g_1 \in G$, hence defines a function of $g_1K \in X$ to which S_1 can be applied (assuming that S_1 or S_2 has compact support). A more classical definition ([9], p. 290) of $S_1 * S_2$ arises from the convolution on the group *G* itself, by means of the projection $G \to G/K$; it is easily checked that both definitions agree, but (1) will be more convenient here (and could be used even if *K* were not compact).

PROPOSITION 3. Let X = G/K with K compact, and assume that Y = G/H has a G-invariant measure. For any $u \in C_c(X)$ we have

$$R^*Ru = u * S,$$

a convolution on X. Here, denoting by δ the Dirac measure at the origin $x_o = K$ of X, the distribution $S = R^*R\delta$ is the K-invariant measure on X given by

$$\langle S, u \rangle = R^* R u(x_o) = \int_{K \times H} u(kh \cdot x_o) \, dk \, dh = R u_K(y_o) \, ,$$

with $u_K(x) = \int_K u(k \cdot x) dk$ and $y_o = H$.

Proof. The definition of the Radon transforms R and R^* clearly show that they intertwine the actions of G on X and Y (here denoted by $\tau_X(g)$, resp. $\tau_Y(g)$, for $g \in G$):

$$R(u \circ \tau_X(g)) = (Ru) \circ \tau_Y(g), \quad R^*(v \circ \tau_Y(g)) = (R^*v) \circ \tau_X(g).$$

Therefore R^*R commutes with $\tau_X(g)$, hence is a right convolution operator. Indeed, let $\varphi \in \mathcal{D}(X)$ be a test function. The distribution S defined by $\langle S, \varphi \rangle = R^* R \varphi(x_o)$ extends to a K-invariant positive linear form on $C_c(X)$, i.e. a measure, and

The last equality follows from the duality between R and R^* (Proposition 2).

b. TOTALLY GEODESIC TRANSFORM ON ISOTROPIC SPACES. The following variant of Proposition 3 gives a more precise statement in a specific situation. Unifying and extending several results from the literature on totally geodesic Radon transforms on two-point homogeneous spaces (Helgason [9], p. 104, 124 and 160, Berenstein and Casadio Tarabusi [1] p. 618), it will lead to inversion formulas. Let X = G/K be an *isotropic* connected non compact Riemannian manifold with distance d, where G is a transitive Lie group of isometries of X and K is the isotropy subgroup of some origin $x_o \in X$. Let y_o be a *totally geodesic* submanifold of X, containing x_o , and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$. We denote by A(r), resp. $A_o(r)$, the Riemannian measure (area) of a sphere of radius r in X, resp. in y_o .

As explained in Section 4.1 a below, Lemma 1 applies to this situation and the Radon transform can be written as

$$Ru(y) = \int_{y} u(x) \, dm_y(x) \,, \quad u \in C_c(X) \,, \quad y \in Y \,,$$

where dm_y is the Riemannian measure induced by X on its submanifold y, and

$$R^*v(g \cdot x_o) = \int_K v(gk \cdot y_o) \, dk \,, \quad v \in C(Y), \ g \in G \,.$$

Note that we will not need here the group H nor an invariant measure on G/H, as opposed to Proposition 3.

PROPOSITION 4. With the above notation we have, for any $u \in C_c(X)$,

$$R^*Ru = u * S$$

(convolution on X), where S is the K-invariant function on X defined by

$$S(x) = A_o(r)/A(r), \quad r = d(x_o, x).$$

An explicit formula (4) for S will be given in Section 4.1, after we introduce the relevant notations.

Proof. Fix $z = g \cdot x_o \in X$. The measure dm_y on $y = gk \cdot y_o$ corresponds to the measure dm_o on y_o by the isometry $x \mapsto gk \cdot x$, whence

$$R^*Ru(z) = \int_{y_o} \left(\int_K u(gk \cdot x) \, dk \right) \, dm_o(x) \, dx$$

Now, X being isotropic, K-orbits are spheres centered at x_o . Since $\int_K dk = 1$, the above integral over K is the mean value $(M_r u)(z)$ of u over the sphere $\Sigma(z, r)$ with center z and radius $r = d(x_o, x)$. Therefore

$$\int_{K} u(gk \cdot x) \, dk = (M_{r}u)(z) = \frac{1}{A(r)} \int_{\Sigma(z,r)} u \, d\sigma$$

where $d\sigma$ is the Riemannian measure on $\Sigma(z, r)$, and

$$R^*Ru(z) = \int_{y_o} (M_r u)(z) \, dm_o(x) \, .$$

But, y_o being totally geodesic, the distance $r = d(x_o, x)$ between two points of y_o is the same in X and in y_o , and the latter integral can thus be computed in geodesic polar coordinates on y_o (with center x_o), as

$$R^*Ru(z) = \int_0^\infty (M_r u)(z) A_o(r) dr$$
$$= \int_0^\infty (M_r u)(z) A(r) f(r) dr$$

with $f(r) = A_o(r)/A(r)$. This in turn can be viewed as an integral over X computed in polar coordinates (with center z), namely

$$R^*Ru(z) = \int_0^\infty f(r) \, dr \int_{\Sigma(z,r)} u \, d\sigma = \int_X u(x) f(d(z,x)) \, dx \, d\sigma$$

Setting $z = g \cdot x_o$, $x = g' \cdot x_o$ it follows that, for any test function $\varphi \in \mathcal{D}(X)$,

$$\int_X R^* Ru(z) \varphi(z) dz = \int_{G \times G} u(g' \cdot x_o) f(d(g \cdot x_o, g' \cdot x_o)) \varphi(g \cdot x_o) dg' dg.$$

Changing the variable g into g = g'g'' (with fixed g') in $\int dg$, we obtain from the left invariance of dg

$$\int_{X} R^* Ru(z) \varphi(z) dz = \int_{G \times G} u(g' \cdot x_o) f(d(g'' \cdot x_o, x_o)) \varphi(g'g'' \cdot x_o) dg' dg''$$
$$= \langle u * S, \varphi \rangle,$$

according to (1) and the definition of S in the proposition.

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c. HOROCYCLE TRANSFORM ON RANK ONE SPACES. Let X = G/K be a Riemannian symmetric space of the noncompact type, G = KAN an Iwasawa decomposition (cf. Notations, d) and Y = G/MN the space of all horocycles in X. The corresponding dual Radon transforms are

$$Ru(gMN) = \int_{N} u(gnK) \, dn \, , \quad R^* v(gK) = \int_{K} v(gkN) \, dk$$

for $u \in C_c(X)$, $v \in C(Y)$; MN has been replaced by N in the right-hand sides because K contains M.

We now specialize to rank one spaces, with positive roots α and (possibly) 2α . Let *H* be the basis vector of \mathfrak{a} such that $\alpha(H) = 1$. Multiplying the Killing form scalar product on \mathfrak{g} by a suitable factor, it will be convenient to assume that the corresponding norm on \mathfrak{p} satisfies ||H|| = 1.

The exponential mapping $\exp : \mathfrak{n} = \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha} \to N$ is a diffeomorphism onto, with Jacobian 1; the Haar measure dn on N can therefore be chosen so that

$$\int_N f(n) \, dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} f(\exp(Z+T)) \, dZ dT \,,$$

where dZ, resp. dT, is the Lebesgue measure on \mathfrak{g}_{α} , resp. $\mathfrak{g}_{2\alpha}$, corresponding to the norm $\| \|$.

Let $p = \dim \mathfrak{g}_{\alpha}$, $q = \dim \mathfrak{g}_{2\alpha}$, $\rho = (p/2) + q$, $n = p + q + 1 = \dim X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. With the above normalizations we now have the following analogue of Proposition 4.

PROPOSITION 5. For the horocycle Radon transform on X, a rank one Riemannian symmetric space of the noncompact type, and $u \in C_c(X)$ we have

$$R^*Ru = u * S,$$

(convolution on X). Here S is the radial function on X given by $\dot{}$

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right),$$

with r > 0. For $X = H^n(\mathbf{R})$, i.e. q = 0, this reduces to

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} \left(\cosh \frac{r}{2}\right)^{3-n} .$$

Proof. We first assume q = 0. The groups G and MN being unimodular, the space Y = G/MN has a G-invariant measure ([11], p. 100). By Proposition 3 it follows that $R^*Ru = u * S$, with

$$\langle S, u \rangle = \int_N u(n \cdot x_o) \, dn = \int_{\mathfrak{g}_\alpha} u(\exp Z \cdot x_o) \, dZ$$

for any K-invariant function u on X (this will suffice to find the K-invariant function S).

By classical rank one computations ([8], p. 414), the radial component $\exp(rH)$ of $\exp Z$ is given by

$$\exp Z \cdot x_o = k \exp(rH) \cdot x_o \,,$$

with $k \in K$, $r \ge 0$ and $||Z|| = 2\sqrt{2}\sinh(r/2)$. Using spherical coordinates in $\mathfrak{g}_{\alpha} = \mathbf{R}^{n-1}$ it follows that, for K-invariant u,

$$\int_N u(n \cdot x_o) \, dn = \int_0^\infty u(\operatorname{Exp} rH) f(r) \, dr \, ,$$

with

$$f(r) = 2^{(3/2)(n-1)-1} \omega_{n-1} \left(\sinh \frac{r}{2}\right)^{n-2} \cosh \frac{r}{2}.$$

On the other hand, using the diffeomorphism Exp and spherical coordinates on \mathfrak{p} we have

$$\int_X u(x) \, dx = \int_0^\infty u(\operatorname{Exp} \, rH) A(r) \, dr \, \text{, with } A(r) = \omega_n (\sinh r)^{n-1}$$

(cf. Section 4.1 **b** for more details). If S(r) = f(r)/A(r) we thus have, for *K*-invariant *u*,

$$\int_{N} u(n \cdot x_o) dn = \int_{0}^{\infty} u(\operatorname{Exp} rH) S(r) A(r) dr = \int_{X} u(x) S(x) dx,$$

as claimed.

The case $q \ge 1$ will not be used in the sequel; we sketch its proof, similar to that of the case q = 0. First

$$\langle S, u \rangle = \int_N u(n \cdot x_o) \, dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} u(\exp(Z+T) \cdot x_o) \, dZ \, dT \, .$$

Then, by rank one computations ([8], p. 414),

$$\exp(Z+T) \cdot x_o = k \exp(rH) \cdot x_o, \quad k \in K,$$

$$\cosh^2 r = \left(1 + \frac{1}{4} ||Z||^2\right)^2 + \frac{1}{2} ||T||^2, \quad r \ge 0.$$

Let $x = ||Z||^2/4$, $y = ||T||^2/2$. Using spherical coordinates in $\mathfrak{g}_{\alpha} = \mathbf{R}^p$ and $\mathfrak{g}_{2\alpha} = \mathbf{R}^q$ we obtain

$$\int_{N} u(n \cdot x_{o}) dn = 2^{p-2+(q/2)} \omega_{p} \omega_{q} \int_{0}^{\infty} \int_{0}^{\infty} u(\exp(rH) \cdot x_{o}) x^{(p/2)-1} y^{(q/2)-1} dx dy$$
$$= \int_{0}^{\infty} u(\exp(rH) \cdot x_{o}) f(r) dr.$$

The latter expression follows from the change of variables $(x, r) \mapsto (x, y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)}\omega_p\omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1} \left(\cosh^2 r - (1+x)^2\right)^{(q/2)-1} dx.$$

Setting $x = t(\cosh r - 1)$ we find

$$\begin{split} f(r) &= 2^{(3p+q)/2} \omega_{n-1} \left(\sinh r \right)^{q-1} \left(\sinh \frac{r}{2} \right)^p \cosh r \\ &\times \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \int_0^1 t^{(p/2)-1} (1-t)^{(q/2)-1} \left(1+t \tanh^2 \frac{r}{2} \right)^{(q/2)-1} dt \\ &= 2^{(3p+q)/2} \omega_{n-1} \left(\sinh r \right)^{q-1} \left(\sinh \frac{r}{2} \right)^p \cosh r \\ &\times {}_2F_1 \left(\frac{p}{2}, 1-\frac{q}{2}; \frac{p+q}{2}; -\tanh^2 \frac{r}{2} \right), \end{split}$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for $_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2} \omega_{n-1} (\sinh r)^{n-2} (\cosh r)^q {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right) \,.$$

Thus, for K-invariant u,

$$\int_N u(n \cdot x_o) \, dn = \int_0^\infty u\left(\exp(rH) \cdot x_o\right) \, S(r) A(r) \, dr = \int_X u(x) \, S(x) \, dx \, ,$$

where $A(r) = \omega_n (\sinh r)^{n-1} (\cosh r)^q$ and S(r) = f(r)/A(r).

3.2 RADON INVERSION BY CONVOLUTION

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u * S = R^*Ru$, in the form

$$(2) u = DR^*Ru.$$

To recover u(x) from Ru the recipe will be to integrate Ru(y) over all y incident to x, and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, R^*R commutes with the action of G on X, and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S * T = \delta$, then

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$$u = (R^*Ru) * T.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G-invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality u = u * DS = D(u * S). Indeed, for any test function φ ,

$$\begin{split} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^{t} D\varphi \rangle \\ &= \langle u(g \cdot x_{o}), \langle S, ({}^{t} D\varphi) \circ \tau(g) \rangle \rangle \qquad \text{by (1)} \\ &= \langle u(g \cdot x_{o}), \langle S, {}^{t} D(\varphi \circ \tau(g)) \rangle \rangle \,, \end{split}$$

since the transpose operator ${}^{t}D$ is *G*-invariant too, as follows from the existence of a *G*-invariant measure on *X*. Finally,

$$\langle D(u * S), \varphi \rangle = \left\langle u(g \cdot x_o), \langle DS, \varphi \circ \tau(g) \rangle \right\rangle \\ = \left\langle u * DS, \varphi \right\rangle,$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can find a G-invariant differential operator D on X with given fundamental solution S. We shall now investigate this question on the basis of Propositions 4 and 5.

4. RADON TRANSFORMS ON ISOTROPIC SPACES

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^{n}$$
 or $H^{m}(\mathbf{R}), H^{2m}(\mathbf{C}), H^{4m}(\mathbf{H}), H^{16}(\mathbf{O}),$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the *d*-geodesic Radon transform on *X*, defined by integrating over a family of *d*-dimensional totally geodesic submanifolds of *X*. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 TOTALLY GEODESIC SUBMANIFOLDS

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.