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3. CONVOLUTION ON X AND INVERSION OF R

3.1 A CONVOLUTION FORMULA

Again G is a Lie group, K a compact subgroup, $X = G/K$ and $\tau(g)$ denotes the natural action of G on X , i.e. $\tau(g)x = g \cdot x$.

a. A GENERAL RESULT. Let $S_1, S_2 \in \mathcal{D}'(X)$ be two distributions on X , with S_2 assumed K -invariant. By analogy with the group case (if K were the trivial subgroup), the convolution $S_1 * S_2 \in \mathcal{D}'(X)$ can be defined by

$$(1) \quad \begin{aligned} \langle S_1 * S_2, \varphi \rangle &= \langle S_1(g_1K), \langle S_2(g_2K), \varphi(g_1g_2K) \rangle \rangle \\ &= \langle S_1(g_1K), \langle S_2, \varphi \circ \tau(g_1) \rangle \rangle, \end{aligned}$$

for any $\varphi \in \mathcal{D}(X)$. Indeed, the K -invariance of S_2 implies that $\langle S_2, \varphi \circ \tau(g_1) \rangle$ is a right K -invariant function of $g_1 \in G$, hence defines a function of $g_1K \in X$ to which S_1 can be applied (assuming that S_1 or S_2 has compact support). A more classical definition ([9], p. 290) of $S_1 * S_2$ arises from the convolution on the group G itself, by means of the projection $G \rightarrow G/K$; it is easily checked that both definitions agree, but (1) will be more convenient here (and could be used even if K were not compact).

PROPOSITION 3. Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G -invariant measure. For any $u \in C_c(X)$ we have

$$R^*Ru = u * S,$$

a convolution on X . Here, denoting by δ the Dirac measure at the origin $x_o = K$ of X , the distribution $S = R^*R\delta$ is the K -invariant measure on X given by

$$\langle S, u \rangle = R^*Ru(x_o) = \int_{K \times H} u(kh \cdot x_o) dk dh = Ru_K(y_o),$$

with $u_K(x) = \int_K u(k \cdot x) dk$ and $y_o = H$.

Proof. The definition of the Radon transforms R and R^* clearly show that they intertwine the actions of G on X and Y (here denoted by $\tau_X(g)$, resp. $\tau_Y(g)$, for $g \in G$):

$$R(u \circ \tau_X(g)) = (Ru) \circ \tau_Y(g), \quad R^*(v \circ \tau_Y(g)) = (R^*v) \circ \tau_X(g).$$

Therefore R^*R commutes with $\tau_X(g)$, hence is a right convolution operator. Indeed, let $\varphi \in \mathcal{D}(X)$ be a test function. The distribution S defined by

$\langle S, \varphi \rangle = R^*R\varphi(x_0)$ extends to a K -invariant positive linear form on $C_c(X)$, i.e. a measure, and

$$\begin{aligned} \langle u * S, \varphi \rangle &= \langle u(g \cdot x_0), \langle S, \varphi \circ \tau_X(g) \rangle \rangle && \text{by (1)} \\ &= \langle u(g \cdot x_0), R^*R(\varphi \circ \tau_X(g))(x_0) \rangle \\ &= \langle u(g \cdot x_0), (R^*R\varphi)(g \cdot x_0) \rangle \\ &= \langle u, R^*R\varphi \rangle = \langle R^*Ru, \varphi \rangle. \end{aligned}$$

The last equality follows from the duality between R and R^* (Proposition 2). \square

b. TOTALLY GEODESIC TRANSFORM ON ISOTROPIC SPACES. The following variant of Proposition 3 gives a more precise statement in a specific situation. Unifying and extending several results from the literature on totally geodesic Radon transforms on two-point homogeneous spaces (Helgason [9], p.104, 124 and 160, Berenstein and Casadio Tarabusi [1] p.618), it will lead to inversion formulas. Let $X = G/K$ be an *isotropic* connected non compact Riemannian manifold with distance d , where G is a transitive Lie group of isometries of X and K is the isotropy subgroup of some origin $x_0 \in X$. Let y_0 be a *totally geodesic* submanifold of X , containing x_0 , and let Y be the set of all submanifolds $y = g \cdot y_0$ of X , with $g \in G$. We denote by $A(r)$, resp. $A_o(r)$, the Riemannian measure (area) of a sphere of radius r in X , resp. in y_0 .

As explained in Section 4.1 a below, Lemma 1 applies to this situation and the Radon transform can be written as

$$Ru(y) = \int_y u(x) dm_y(x), \quad u \in C_c(X), \quad y \in Y,$$

where dm_y is the Riemannian measure induced by X on its submanifold y , and

$$R^*v(g \cdot x_0) = \int_K v(gk \cdot y_0) dk, \quad v \in C(Y), \quad g \in G.$$

Note that we will not need here the group H nor an invariant measure on G/H , as opposed to Proposition 3.

PROPOSITION 4. *With the above notation we have, for any $u \in C_c(X)$,*

$$R^*Ru = u * S$$

(convolution on X), where S is the K -invariant function on X defined by

$$S(x) = A_o(r)/A(r), \quad r = d(x_0, x).$$

An explicit formula (4) for S will be given in Section 4.1, after we introduce the relevant notations.

Proof. Fix $z = g \cdot x_o \in X$. The measure dm_y on $y = gk \cdot y_o$ corresponds to the measure dm_o on y_o by the isometry $x \mapsto gk \cdot x$, whence

$$R^*Ru(z) = \int_{y_o} \left(\int_K u(gk \cdot x) dk \right) dm_o(x).$$

Now, X being isotropic, K -orbits are spheres centered at x_o . Since $\int_K dk = 1$, the above integral over K is the mean value $(M_ru)(z)$ of u over the sphere $\Sigma(z, r)$ with center z and radius $r = d(x_o, x)$. Therefore

$$\int_K u(gk \cdot x) dk = (M_ru)(z) = \frac{1}{A(r)} \int_{\Sigma(z,r)} u d\sigma,$$

where $d\sigma$ is the Riemannian measure on $\Sigma(z, r)$, and

$$R^*Ru(z) = \int_{y_o} (M_ru)(z) dm_o(x).$$

But, y_o being totally geodesic, the distance $r = d(x_o, x)$ between two points of y_o is the same in X and in y_o , and the latter integral can thus be computed in geodesic polar coordinates on y_o (with center x_o), as

$$\begin{aligned} R^*Ru(z) &= \int_0^\infty (M_ru)(z) A_o(r) dr \\ &= \int_0^\infty (M_ru)(z) A(r) f(r) dr \end{aligned}$$

with $f(r) = A_o(r)/A(r)$. This in turn can be viewed as an integral over X computed in polar coordinates (with center z), namely

$$R^*Ru(z) = \int_0^\infty f(r) dr \int_{\Sigma(z,r)} u d\sigma = \int_X u(x) f(d(z, x)) dx.$$

Setting $z = g \cdot x_o$, $x = g' \cdot x_o$ it follows that, for any test function $\varphi \in \mathcal{D}(X)$,

$$\int_X R^*Ru(z) \varphi(z) dz = \int_{G \times G} u(g' \cdot x_o) f(d(g \cdot x_o, g' \cdot x_o)) \varphi(g \cdot x_o) dg' dg.$$

Changing the variable g into $g = g'g''$ (with fixed g') in $\int dg$, we obtain from the left invariance of dg

$$\begin{aligned} \int_X R^*Ru(z) \varphi(z) dz &= \int_{G \times G} u(g' \cdot x_o) f(d(g'' \cdot x_o, x_o)) \varphi(g'g'' \cdot x_o) dg' dg'' \\ &= \langle u * S, \varphi \rangle, \end{aligned}$$

according to (1) and the definition of S in the proposition. □

c. HOROCYCLE TRANSFORM ON RANK ONE SPACES. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type, $G = KAN$ an Iwasawa decomposition (cf. Notations, d) and $Y = G/MN$ the space of all horocycles in X . The corresponding dual Radon transforms are

$$Ru(gMN) = \int_N u(gnK) dn, \quad R^*v(gK) = \int_K v(gkN) dk$$

for $u \in C_c(X)$, $v \in C(Y)$; MN has been replaced by N in the right-hand sides because K contains M .

We now specialize to rank one spaces, with positive roots α and (possibly) 2α . Let H be the basis vector of \mathfrak{a} such that $\alpha(H) = 1$. Multiplying the Killing form scalar product on \mathfrak{g} by a suitable factor, it will be convenient to assume that the corresponding norm on \mathfrak{p} satisfies $\|H\| = 1$.

The exponential mapping $\exp : \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \rightarrow N$ is a diffeomorphism onto, with Jacobian 1; the Haar measure dn on N can therefore be chosen so that

$$\int_N f(n) dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} f(\exp(Z + T)) dZdT,$$

where dZ , resp. dT , is the Lebesgue measure on \mathfrak{g}_α , resp. $\mathfrak{g}_{2\alpha}$, corresponding to the norm $\|\cdot\|$.

Let $p = \dim \mathfrak{g}_\alpha$, $q = \dim \mathfrak{g}_{2\alpha}$, $\rho = (p/2) + q$, $n = p + q + 1 = \dim X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. With the above normalizations we now have the following analogue of Proposition 4.

PROPOSITION 5. *For the horocycle Radon transform on X , a rank one Riemannian symmetric space of the noncompact type, and $u \in C_c(X)$ we have*

$$R^*Ru = u * S,$$

(convolution on X). Here S is the radial function on X given by

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right),$$

with $r > 0$. For $X = H^n(\mathbf{R})$, i.e. $q = 0$, this reduces to

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} \left(\cosh \frac{r}{2}\right)^{3-n}.$$

Proof. We first assume $q = 0$. The groups G and MN being unimodular, the space $Y = G/MN$ has a G -invariant measure ([11], p.100). By Proposition 3 it follows that $R^*Ru = u * S$, with

$$\langle S, u \rangle = \int_N u(n \cdot x_o) dn = \int_{\mathfrak{g}_\alpha} u(\exp Z \cdot x_o) dZ$$

for any K -invariant function u on X (this will suffice to find the K -invariant function S).

By classical rank one computations ([8], p.414), the radial component $\exp(rH)$ of $\exp Z$ is given by

$$\exp Z \cdot x_o = k \exp(rH) \cdot x_o,$$

with $k \in K$, $r \geq 0$ and $\|Z\| = 2\sqrt{2} \sinh(r/2)$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^{n-1}$ it follows that, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\text{Exp } rH) f(r) dr,$$

with

$$f(r) = 2^{(3/2)(n-1)-1} \omega_{n-1} \left(\sinh \frac{r}{2} \right)^{n-2} \cosh \frac{r}{2}.$$

On the other hand, using the diffeomorphism Exp and spherical coordinates on \mathfrak{p} we have

$$\int_X u(x) dx = \int_0^\infty u(\text{Exp } rH) A(r) dr, \text{ with } A(r) = \omega_n (\sinh r)^{n-1}$$

(cf. Section 4.1 **b** for more details). If $S(r) = f(r)/A(r)$ we thus have, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\text{Exp } rH) S(r) A(r) dr = \int_X u(x) S(x) dx,$$

as claimed.

The case $q \geq 1$ will not be used in the sequel; we sketch its proof, similar to that of the case $q = 0$. First

$$\langle S, u \rangle = \int_N u(n \cdot x_o) dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} u(\exp(Z + T) \cdot x_o) dZ dT.$$

Then, by rank one computations ([8], p.414),

$$\begin{aligned} \exp(Z + T) \cdot x_o &= k \exp(rH) \cdot x_o, \quad k \in K, \\ \cosh^2 r &= \left(1 + \frac{1}{4} \|Z\|^2 \right)^2 + \frac{1}{2} \|T\|^2, \quad r \geq 0. \end{aligned}$$

Let $x = \|Z\|^2/4$, $y = \|T\|^2/2$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbf{R}^p$ and $\mathfrak{g}_{2\alpha} = \mathbf{R}^q$ we obtain

$$\begin{aligned} \int_N u(n \cdot x_o) dn &= 2^{p-2+(q/2)} \omega_p \omega_q \int_0^\infty \int_0^\infty u(\exp(rH) \cdot x_o) x^{(p/2)-1} y^{(q/2)-1} dx dy \\ &= \int_0^\infty u(\exp(rH) \cdot x_o) f(r) dr. \end{aligned}$$

The latter expression follows from the change of variables $(x, r) \mapsto (x, y)$, with Jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)} \omega_p \omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1} (\cosh^2 r - (1+x)^2)^{(q/2)-1} dx.$$

Setting $x = t(\cosh r - 1)$ we find

$$\begin{aligned} f(r) &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \int_0^1 t^{(p/2)-1} (1-t)^{(q/2)-1} \left(1 + t \tanh^2 \frac{r}{2}\right)^{(q/2)-1} dt \\ &= 2^{(3p+q)/2} \omega_{n-1} (\sinh r)^{q-1} \left(\sinh \frac{r}{2}\right)^p \cosh r \\ &\quad \times {}_2F_1\left(\frac{p}{2}, 1 - \frac{q}{2}; \frac{p+q}{2}; -\tanh^2 \frac{r}{2}\right), \end{aligned}$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3], p. 113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2} \omega_{n-1} (\sinh r)^{n-2} (\cosh r)^q {}_2F_1\left(\frac{\rho-1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r\right).$$

Thus, for K -invariant u ,

$$\int_N u(n \cdot x_o) dn = \int_0^\infty u(\exp(rH) \cdot x_o) S(r) A(r) dr = \int_X u(x) S(x) dx,$$

where $A(r) = \omega_n (\sinh r)^{n-1} (\cosh r)^q$ and $S(r) = f(r)/A(r)$. \square

3.2 RADON INVERSION BY CONVOLUTION

Radon inversion formulas will follow from Section 3.1 if we can solve for u the convolution equation $u * S = R^* R u$, in the form

$$(2) \quad u = D R^* R u.$$

To recover $u(x)$ from $R u$ the recipe will be to integrate $R u(y)$ over all y incident to x , and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, $R^* R$ commutes with the action of G on X , and it is natural to look for a D with the same property, i.e. a convolution operator: if T is a distribution on X such that $S * T = \delta$, then

$$u = (R^*Ru) * T.$$

Though the question can be tackled by harmonic analysis on X (cf. Section 5), a G -invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ ,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u * S, {}^tD\varphi \rangle \\ &= \langle u(g \cdot x_0), \langle S, ({}^tD\varphi) \circ \tau(g) \rangle \rangle \quad \text{by (1)} \\ &= \langle u(g \cdot x_0), \langle S, {}^tD(\varphi \circ \tau(g)) \rangle \rangle, \end{aligned}$$

since the transpose operator tD is G -invariant too, as follows from the existence of a G -invariant measure on X . Finally,

$$\begin{aligned} \langle D(u * S), \varphi \rangle &= \langle u(g \cdot x_0), \langle DS, \varphi \circ \tau(g) \rangle \rangle \\ &= \langle u * DS, \varphi \rangle, \end{aligned}$$

as claimed; assuming G unimodular (as in [9], p. 291) is thus unnecessary here.

The method applies whenever we can find a G -invariant differential operator D on X with given fundamental solution S . We shall now investigate this question on the basis of Propositions 4 and 5.

4. RADON TRANSFORMS ON ISOTROPIC SPACES

Throughout this section X will be an isotropic connected noncompact Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbf{R}^n \text{ or } H^m(\mathbf{R}), H^{2m}(\mathbf{C}), H^{4m}(\mathbf{H}), H^{16}(\mathbf{O}),$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18], §8.12). We first try to invert the d -geodesic Radon transform on X , defined by integrating over a family of d -dimensional totally geodesic submanifolds of X . At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbf{R})$.

4.1 TOTALLY GEODESIC SUBMANIFOLDS

Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.