

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS
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Kapitel: 3.1 Intersection multiplicities
DOI: <https://doi.org/10.5169/seals-65426>

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3. MAIN THEOREM

All theorems from Section 2 are consequences of one theorem on the least number of flattenings of a closed polygon in real projective space.

In his remarkable work [3], M. Barner introduced the notion of a *strictly convex* curve in real projective space: this is a smooth closed curve $\gamma \subset \mathbf{RP}^d$ such that for every $(d-1)$ -tuple of points on γ there exists a hyperplane through these points that does not intersect γ at any other points. Barner discovered the following theorem:

A strictly convex curve has at least $d+1$ distinct flattening points.

Recall that a flattening point of a projective space curve is a point at which the osculating hyperplane is stationary; in other words, this is a singularity of the projectively dual curve. In fact, Barner's result is considerably stronger but we shall not dwell on it here – see [15] for an exposition.

Our goal in this section is to provide a discrete version of Barner's theorem. First we need to develop an elementary intersection formalism for polygonal lines.

3.1 INTERSECTION MULTIPLICITIES

Throughout this section we shall look at closed polygons $P \subset \mathbf{RP}^d$ with vertices V_1, \dots, V_n ($n \geq d+1$) in general position. In other words, for every set of vertices V_{i_1}, \dots, V_{i_k} , where $k \leq d+1$, the span of V_{i_1}, \dots, V_{i_k} is $(k-1)$ -dimensional.

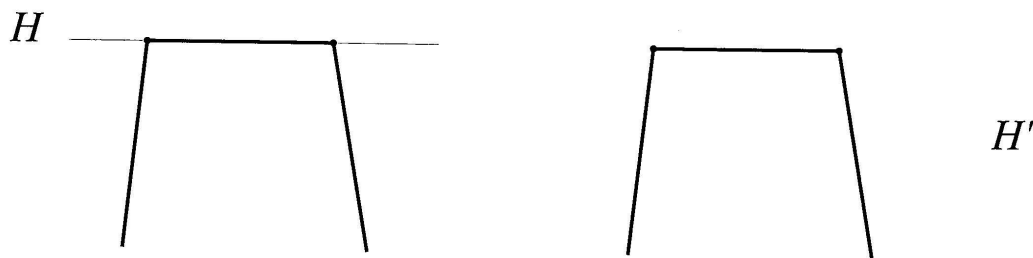
DEFINITION 3.1. A polygon P is said to be *transverse* to a hyperplane H at a point $X \in P \cap H$ if

- (a) X is an interior point of an edge and this edge is transverse to H , or
- (b) X is a vertex, the two edges incident to X are transverse to H and are locally separated by H .

Clearly, transversality is an open condition.

DEFINITION 3.2. A polygon P is said to intersect a hyperplane H with multiplicity k if for every hyperplane H' sufficiently close to H and transverse to P , the number of points $P \cap H'$ does not exceed k and, moreover, k is attained for some H' .

This definition does not exclude the case where a number of vertices of P lie in H .



multiplicity 2

FIGURE 3

LEMMA 3.3. *Let V_{i_1}, \dots, V_{i_k} with $k \leq d$ be vertices of P . Then any hyperplane H passing through V_{i_1}, \dots, V_{i_k} meets P with multiplicity at least k .*

Proof. Move each V_{i_j} ($j = 1, \dots, k$) slightly along the edge $(V_{i_j}, V_{i_{j+1}})$ to obtain a new point V'_{i_j} . Let us show that a generic hyperplane H' through $V'_{i_1}, \dots, V'_{i_k}$ is transverse to P . This will imply the lemma because H' has at least k intersections with P .

It suffices to show that H' does not contain any vertex of P . First we note that, since P is in general position, a generic hyperplane H through V_{i_1}, \dots, V_{i_k} does not contain any other vertex. The same holds true for every hyperplane which is sufficiently close to H . It remains to show that the chosen H' does not contain any of V_{i_1}, \dots, V_{i_k} .

Suppose H' contains V_{i_j} . Then H' contains the edge $(V_{i_j}, V_{i_{j+1}})$ and therefore also $V_{i_{j+1}}$. If $i_j + 1 \notin \{i_1, \dots, i_k\}$ we obtain a contradiction with the previous paragraph. If, on the other hand, $i_j + 1 \in \{i_1, \dots, i_k\}$ then we can proceed in the same way with $V_{i_{j+1}}$. However, we cannot go on indefinitely since $k < n$. \square

The next definition is topological in nature.

DEFINITION 3.4. Consider a continuous curve in \mathbf{RP}^d with endpoints A and Z . Let H be a hyperplane not containing A or Z . We say that A and Z are on one side of H if one can connect A and Z by a curve not intersecting H in such a way that the resulting closed curve is contractible. Otherwise we say that A and Z are separated by H .

Clearly, if one has only two points A and Z (and no curve connecting

them), then one cannot say that the points are on one side of, or separated by, a hyperplane.

LEMMA 3.5. *Let $\Gamma = (A, \dots, Z)$ be a broken line in general position in \mathbf{RP}^d , and let H be a hyperplane not containing A or Z . Denote by k the intersection multiplicity of Γ with H . Then A and Z are separated by H if k is odd and not separated otherwise.*

Proof. Connect Z and A by a segment so as to obtain a closed polygon $\bar{\Gamma}$ and consider a hyperplane H' close to H , transverse to $\bar{\Gamma}$ and intersecting Γ in k points. Since $\bar{\Gamma}$ is contractible, H' intersects $\bar{\Gamma}$ in an even number of points. Therefore, H' intersects the segment (Z, A) for odd k and does not intersect it for even k . \square

The next definition introduces a significant class of polygons which is our main object of study.

DEFINITION 3.6. A polygon P is called *strictly convex* if through every $d - 1$ vertices there passes a hyperplane H whose intersection multiplicity with P is equal to $d - 1$.

This definition becomes, in the smooth limit, that of strict convexity for smooth curves, due to Barner.

DEFINITION 3.7. A d -tuple of consecutive vertices (V_i, \dots, V_{i+d-1}) of a polygon P in \mathbf{RP}^d is called a *flattening* if the endpoints V_{i-1} and V_{i+d} of the broken line $(V_{i-1}, \dots, V_{i+d})$ are:

- (a) separated by the hyperplane through (V_i, \dots, V_{i+d-1}) if d is even,
- (b) not separated if d is odd.

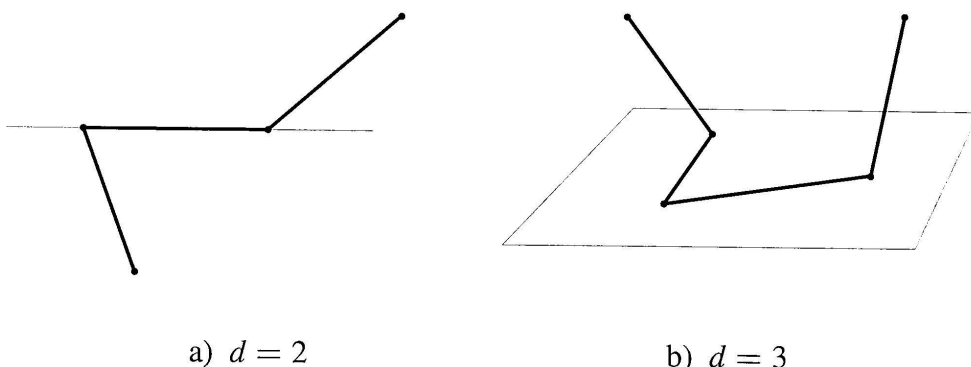


FIGURE 4

REMARK 3.8. A curve in \mathbf{RP}^d can be lifted to $\mathbf{R}^{d+1} \setminus \{0\}$; the lifting is not unique. Given a polygon $P \subset \mathbf{RP}^d$ with vertices V_1, \dots, V_n , we lift it to \mathbf{R}^{d+1} as a polygon \tilde{P} and denote its vertices by $\tilde{V}_1, \dots, \tilde{V}_n$. Then a d -tuple (V_i, \dots, V_{i+d-1}) is a flattening if and only if the determinant

$$(3.1) \quad \Delta_j = |\tilde{V}_j \dots \tilde{V}_{j+d}|$$

changes sign as j varies from $i-1$ to i .

This property is independent of the lifting.

3.2 A SIMPLEX IS STRICTLY CONVEX

Define a simplex $S_d \subset \mathbf{RP}^d$ with vertices V_1, \dots, V_{d+1} as the projection from the punctured \mathbf{R}^{d+1} of the polygonal line:

$$(3.2) \quad \tilde{V}_1 = (1, 0, \dots, 0), \quad \tilde{V}_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad \tilde{V}_{d+1} = (0, \dots, 0, 1)$$

and

$$(3.3) \quad \tilde{V}_{d+2} = (-1)^{d+1} \tilde{V}_1.$$

The last vertex has the same projection as the first one; S_d is contractible for odd d , and non-contractible for even d .

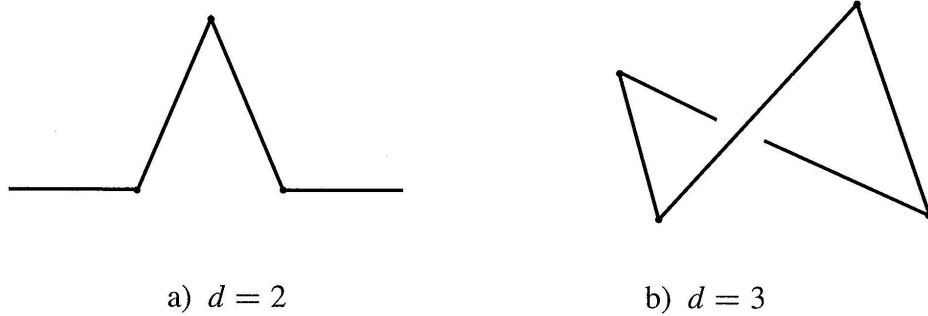


FIGURE 5

PROPOSITION 3.9. *The polygon S_d is strictly convex.*

Proof. We need to prove that through every $(d-1)$ -tuple

$$(V_1, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_{d+1})$$

there passes a hyperplane H intersecting P with multiplicity $d-1$. Select a point W on the line $(\tilde{V}_i, \tilde{V}_j)$ in such a manner that W lies on the segment $(\tilde{V}_i, \tilde{V}_j)$ if $j-i$ is even, and does not lie on it if $j-i$ is odd. Define \tilde{H} as the linear span of $\tilde{V}_1, \dots, \widehat{\tilde{V}_i}, \dots, \widehat{\tilde{V}_j}, \dots, \tilde{V}_{d+1}, W$. We claim that its projection $H \subset \mathbf{RP}^d$ meets S_d with multiplicity $\leq d-1$.