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Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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Kapitel: 3.1 Intersection multiplicities

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3. Main Theorem

All theorems from Section 2 are consequences of one theorem on the least number of flattenings of a closed polygon in real projective space.

In his remarkable work [3], M. Barner introduced the notion of a *strictly convex* curve in real projective space: this is a smooth closed curve $\gamma \subset \mathbf{RP}^d$ such that for every $(d-1)$-tuple of points on γ there exists a hyperplane through these points that does not intersect γ at any other points. Barner discovered the following theorem:

A strictly convex curve has at least $d+1$ distinct flattening points.

Recall that a flattening point of a projective space curve is a point at which the osculating hyperplane is stationary; in other words, this is a singularity of the projectively dual curve. In fact, Barner's result is considerably stronger but we shall not dwell on it here – see [15] for an exposition.

Our goal in this section is to provide a discrete version of Barner's theorem. First we need to develop an elementary intersection formalism for polygonal lines.

3.1 Intersection multiplicities

Throughout this section we shall look at closed polygons $P \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_n$ $(n \geq d+1)$ in general position. In other words, for every set of vertices $V_{i_1}, \ldots, V_{i_k}$, where $k \leq d+1$, the span of $V_{i_1}, \ldots, V_{i_k}$ is $(k-1)$-dimensional.

Definition 3.1. A polygon P is said to be *transverse* to a hyperplane H at a point $X \in P \cap H$ if

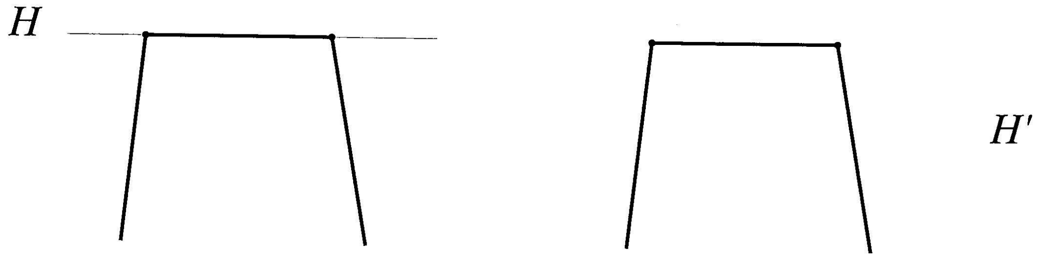
(a) X is an interior point of an edge and this edge is transverse to H, or

(b) X is a vertex, the two edges incident to X are transverse to H and are locally separated by H.

Clearly, transversality is an open condition.

Definition 3.2. A polygon P is said to intersect a hyperplane H with multiplicity k if for every hyperplane H' sufficiently close to H and transverse to P, the number of points $P \cap H'$ does not exceed k and, moreover, k is attained for some H'.

This definition does not exclude the case where a number of vertices of P lie in H.



multiplicity 2

FIGURE 3

LEMMA 3.3. *Let $V_{i_1}, \dots, V_{i_k}$ with $k \leq d$ be vertices of P. Then any hyperplane H passing through $V_{i_1}, \dots, V_{i_k}$ meets P with multiplicity at least k.*

Proof. Move each V_{i_j} $(j = 1, \dots, k)$ slightly along the edge (V_{i_j}, V_{i_j+1}) to obtain a new point V'_{i_j}. Let us show that a generic hyperplane H' through $V'_{i_1}, \dots, V'_{i_k}$ is transverse to P. This will imply the lemma because H' has at least k intersections with P.

It suffices to show that H' does not contain any vertex of P. First we note that, since P is in general position, a generic hyperplane H through $V_{i_1}, \dots, V_{i_k}$ does not contain any other vertex. The same holds true for every hyperplane which is sufficiently close to H. It remains to show that the chosen H' does not contain any of $V_{i_1}, \dots, V_{i_k}$.

Suppose H' contains V_{i_j}. Then H' contains the edge (V_{i_j}, V_{i_j+1}) and therefore also V_{i_j+1}. If $i_j + 1 \notin \{i_1, \dots, i_k\}$ we obtain a contradiction with the previous paragraph. If, on the other hand, $i_j + 1 \in \{i_1, \dots, i_k\}$ then we can proceed in the same way with V_{i_j+1}. However, we cannot go on indefinitely since $k < n$. $\square$

The next definition is topological in nature.

DEFINITION 3.4. Consider a continuous curve in $\mathbf{RP}^d$ with endpoints A and Z. Let H be a hyperplane not containing A or Z. We say that A and Z *are on one side of* H if one can connect A and Z by a curve not intersecting H in such a way that the resulting closed curve is contractible. Otherwise we say that A and Z are *separated by* H.

Clearly, if one has only two points A and Z (and no curve connecting them), then one cannot say that the points are on one side of, or separated by, a hyperplane.

LEMMA 3.5. *Let $\Gamma = (A, \dots, Z)$ be a broken line in general position in $\mathbf{RP}^d$, and let H be a hyperplane not containing A or Z. Denote by k the intersection multiplicity of Γ with H. Then A and Z are separated by H if k is odd and not separated otherwise.*

Proof. Connect Z and A by a segment so as to obtain a closed polygon $\overline{\Gamma}$ and consider a hyperplane H' close to H, transverse to $\overline{\Gamma}$ and intersecting Γ in k points. Since $\overline{\Gamma}$ is contractible, H' intersects $\overline{\Gamma}$ in an even number of points. Therefore, H' intersects the segment (Z, A) for odd k and does not intersect it for even k. □

The next definition introduces a significant class of polygons which is our main object of study.

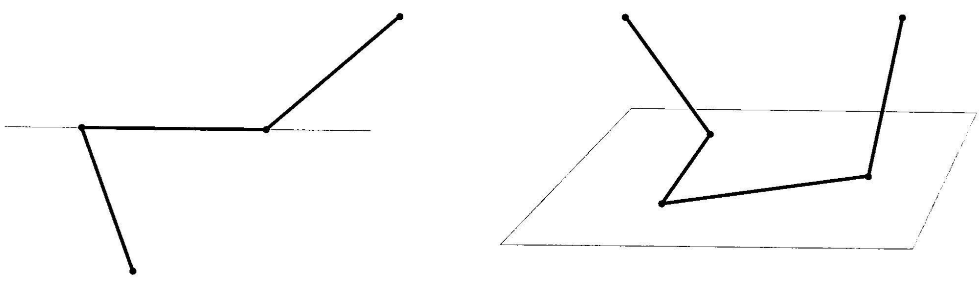
DEFINITION 3.6. A polygon P is called *strictly convex* if through every $d-1$ vertices there passes a hyperplane H whose intersection multiplicity with P is equal to $d-1$.

This definition becomes, in the smooth limit, that of strict convexity for smooth curves, due to Barner.

DEFINITION 3.7. A d-tuple of consecutive vertices $(V_i, \dots, V_{i+d-1})$ of a polygon P in $\mathbf{RP}^d$ is called a *flattening* if the endpoints V_{i-1} and V_{i+d} of the broken line $(V_{i-1}, \dots, V_{i+d})$ are:

(a) separated by the hyperplane through $(V_i, \dots, V_{i+d-1})$ if d is even,

(b) not separated if d is odd.



a) $d = 2$ b) $d = 3$

FIGURE 4

REMARK 3.8. A curve in $\mathbf{RP}^d$ can be lifted to $\mathbf{R}^{d+1} \setminus \{0\}$; the lifting is not unique. Given a polygon $P \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_n$, we lift it to $\mathbf{R}^{d+1}$ as a polygon $\widetilde{P}$ and denote its vertices by $\widetilde{V}_1, \ldots \widetilde{V}_n$. Then a d-tuple $(V_i, \ldots, V_{i+d-1})$ is a flattening if and only if the determinant

$$\Delta_j = \left|\widetilde{V}_j \ldots \widetilde{V}_{j+d}\right| \tag{3.1}$$

changes sign as j varies from $i-1$ to i.

This property is independent of the lifting.

3.2 A SIMPLEX IS STRICTLY CONVEX

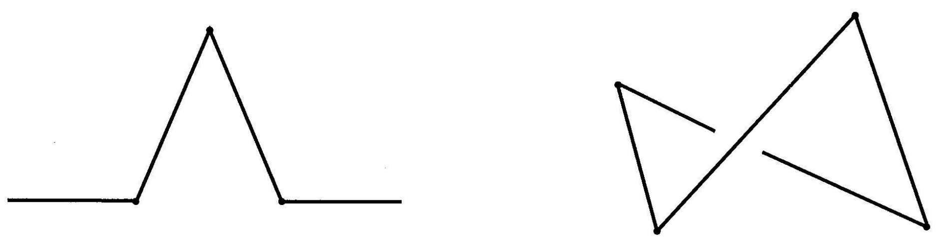
Define a simplex $S_d \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_{d+1}$ as the projection from the punctured $\mathbf{R}^{d+1}$ of the polygonal line:

$$\widetilde{V}_1 = (1, 0, \ldots, 0), \quad \widetilde{V}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \widetilde{V}_{d+1} = (0, \ldots, 0, 1) \tag{3.2}$$

and

$$\widetilde{V}_{d+2} = (-1)^{d+1} \widetilde{V}_1 . \tag{3.3}$$

The last vertex has the same projection as the first one; S_d is contractible for odd d, and non-contractible for even d.



a) $d = 2$ b) $d = 3$

FIGURE 5

PROPOSITION 3.9. *The polygon S_d is strictly convex.*

Proof. We need to prove that through every $(d-1)$-tuple

$$(V_1, \ldots, \widehat{V}_i, \ldots, \widehat{V}_j, \ldots, V_{d+1})$$

there passes a hyperplane H intersecting P with multiplicity $d-1$. Select a point W on the line $(\widetilde{V}_i, \widetilde{V}_j)$ in such a manner that W lies on the segment $(\widetilde{V}_i, \widetilde{V}_j)$ if $j-i$ is even, and does not lie on it if $j-i$ is odd. Define $\widetilde{H}$ as the linear span of $\widetilde{V}_1, \ldots, \widehat{\widetilde{V}}_i, \ldots, \widehat{\widetilde{V}}_j, \ldots, \widetilde{V}_{d+1}, W$. We claim that its projection $H \subset \mathbf{RP}^d$ meets S_d with multiplicity $\leq d-1$.