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<b>Autor:</b>	Ovsienko, V / Tabachnikov, S.
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### 3. MAIN THEOREM

All theorems from Section 2 are consequences of one theorem on the least number of flattenings of a closed polygon in real projective space.

In his remarkable work [3], M. Barner introduced the notion of a *strictly convex* curve in real projective space: this is a smooth closed curve  $\gamma \subset \mathbf{RP}^d$  such that for every  $(d - 1)$ -tuple of points on  $\gamma$  there exists a hyperplane through these points that does not intersect  $\gamma$  at any other points. Barner discovered the following theorem:

*A strictly convex curve has at least  $d + 1$  distinct flattening points.*

Recall that a flattening point of a projective space curve is a point at which the osculating hyperplane is stationary; in other words, this is a singularity of the projectively dual curve. In fact, Barner's result is considerably stronger but we shall not dwell on it here – see [15] for an exposition.

Our goal in this section is to provide a discrete version of Barner's theorem. First we need to develop an elementary intersection formalism for polygonal lines.

#### 3.1 INTERSECTION MULTIPLICITIES

Throughout this section we shall look at closed polygons  $P \subset \mathbf{RP}^d$  with vertices  $V_1, \dots, V_n$  ( $n \geq d + 1$ ) in general position. In other words, for every set of vertices  $V_{i_1}, \dots, V_{i_k}$ , where  $k \leq d + 1$ , the span of  $V_{i_1}, \dots, V_{i_k}$  is  $(k - 1)$ -dimensional.

**DEFINITION 3.1.** A polygon  $P$  is said to be *transverse* to a hyperplane  $H$  at a point  $X \in P \cap H$  if

- (a)  $X$  is an interior point of an edge and this edge is transverse to  $H$ , or
- (b)  $X$  is a vertex, the two edges incident to  $X$  are transverse to  $H$  and are locally separated by  $H$ .

Clearly, transversality is an open condition.

**DEFINITION 3.2.** A polygon  $P$  is said to intersect a hyperplane  $H$  with multiplicity  $k$  if for every hyperplane  $H'$  sufficiently close to  $H$  and transverse to  $P$ , the number of points  $P \cap H'$  does not exceed  $k$  and, moreover,  $k$  is attained for some  $H'$ .

This definition does not exclude the case where a number of vertices of  $P$  lie in  $H$ .

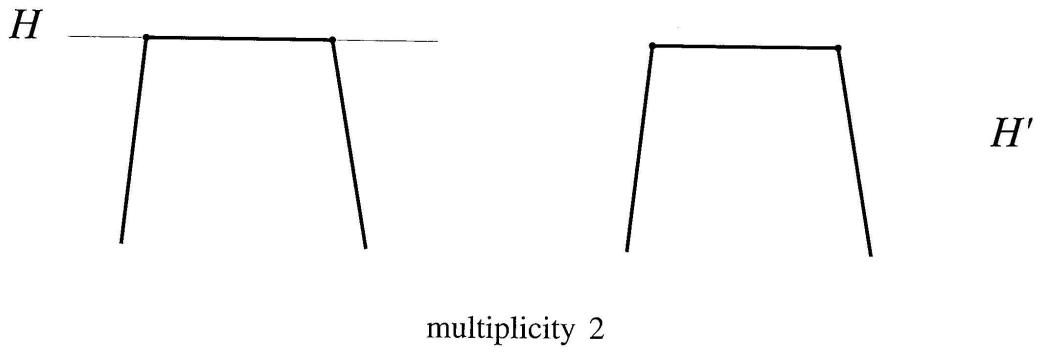


FIGURE 3

LEMMA 3.3. *Let  $V_{i_1}, \dots, V_{i_k}$  with  $k \leq d$  be vertices of  $P$ . Then any hyperplane  $H$  passing through  $V_{i_1}, \dots, V_{i_k}$  meets  $P$  with multiplicity at least  $k$ .*

*Proof.* Move each  $V_{i_j}$  ( $j = 1, \dots, k$ ) slightly along the edge  $(V_{i_j}, V_{i_j+1})$  to obtain a new point  $V'_{i_j}$ . Let us show that a generic hyperplane  $H'$  through  $V'_{i_1}, \dots, V'_{i_k}$  is transverse to  $P$ . This will imply the lemma because  $H'$  has at least  $k$  intersections with  $P$ .

It suffices to show that  $H'$  does not contain any vertex of  $P$ . First we note that, since  $P$  is in general position, a generic hyperplane  $H$  through  $V_{i_1}, \dots, V_{i_k}$  does not contain any other vertex. The same holds true for every hyperplane which is sufficiently close to  $H$ . It remains to show that the chosen  $H'$  does not contain any of  $V_{i_1}, \dots, V_{i_k}$ .

Suppose  $H'$  contains  $V_{i_j}$ . Then  $H'$  contains the edge  $(V_{i_j}, V_{i_j+1})$  and therefore also  $V_{i_j+1}$ . If  $i_j+1 \notin \{i_1, \dots, i_k\}$  we obtain a contradiction with the previous paragraph. If, on the other hand,  $i_j+1 \in \{i_1, \dots, i_k\}$  then we can proceed in the same way with  $V_{i_j+1}$ . However, we cannot go on indefinitely since  $k < n$ .  $\square$

The next definition is topological in nature.

DEFINITION 3.4. Consider a continuous curve in  $\mathbf{RP}^d$  with endpoints  $A$  and  $Z$ . Let  $H$  be a hyperplane not containing  $A$  or  $Z$ . We say that  $A$  and  $Z$  are *on one side of  $H$*  if one can connect  $A$  and  $Z$  by a curve not intersecting  $H$  in such a way that the resulting closed curve is contractible. Otherwise we say that  $A$  and  $Z$  are *separated by  $H$* .

Clearly, if one has only two points  $A$  and  $Z$  (and no curve connecting

them), then one cannot say that the points are on one side of, or separated by, a hyperplane.

LEMMA 3.5. *Let  $\Gamma = (A, \dots, Z)$  be a broken line in general position in  $\mathbf{RP}^d$ , and let  $H$  be a hyperplane not containing  $A$  or  $Z$ . Denote by  $k$  the intersection multiplicity of  $\Gamma$  with  $H$ . Then  $A$  and  $Z$  are separated by  $H$  if  $k$  is odd and not separated otherwise.*

*Proof.* Connect  $Z$  and  $A$  by a segment so as to obtain a closed polygon  $\bar{\Gamma}$  and consider a hyperplane  $H'$  close to  $H$ , transverse to  $\bar{\Gamma}$  and intersecting  $\Gamma$  in  $k$  points. Since  $\bar{\Gamma}$  is contractible,  $H'$  intersects  $\bar{\Gamma}$  in an even number of points. Therefore,  $H'$  intersects the segment  $(Z, A)$  for odd  $k$  and does not intersect it for even  $k$ .  $\square$

The next definition introduces a significant class of polygons which is our main object of study.

DEFINITION 3.6. A polygon  $P$  is called *strictly convex* if through every  $d - 1$  vertices there passes a hyperplane  $H$  whose intersection multiplicity with  $P$  is equal to  $d - 1$ .

This definition becomes, in the smooth limit, that of strict convexity for smooth curves, due to Barner.

DEFINITION 3.7. A  $d$ -tuple of consecutive vertices  $(V_i, \dots, V_{i+d-1})$  of a polygon  $P$  in  $\mathbf{RP}^d$  is called a *flattening* if the endpoints  $V_{i-1}$  and  $V_{i+d}$  of the broken line  $(V_{i-1}, \dots, V_{i+d})$  are:

- (a) separated by the hyperplane through  $(V_i, \dots, V_{i+d-1})$  if  $d$  is even,
- (b) not separated if  $d$  is odd.

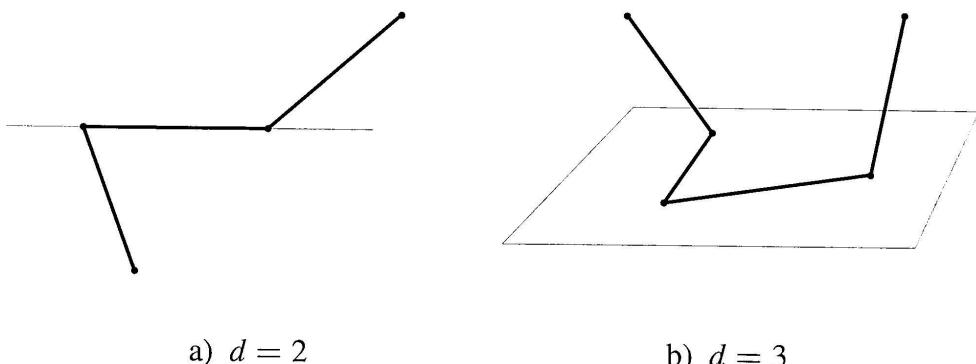


FIGURE 4

REMARK 3.8. A curve in  $\mathbf{RP}^d$  can be lifted to  $\mathbf{R}^{d+1} \setminus \{0\}$ ; the lifting is not unique. Given a polygon  $P \subset \mathbf{RP}^d$  with vertices  $V_1, \dots, V_n$ , we lift it to  $\mathbf{R}^{d+1}$  as a polygon  $\tilde{P}$  and denote its vertices by  $\tilde{V}_1, \dots, \tilde{V}_n$ . Then a  $d$ -tuple  $(V_i, \dots, V_{i+d-1})$  is a flattening if and only if the determinant

$$(3.1) \quad \Delta_j = |\tilde{V}_j \dots \tilde{V}_{j+d}|$$

changes sign as  $j$  varies from  $i-1$  to  $i$ .

This property is independent of the lifting.

### 3.2 A SIMPLEX IS STRICTLY CONVEX

Define a simplex  $S_d \subset \mathbf{RP}^d$  with vertices  $V_1, \dots, V_{d+1}$  as the projection from the punctured  $\mathbf{R}^{d+1}$  of the polygonal line:

$$(3.2) \quad \tilde{V}_1 = (1, 0, \dots, 0), \quad \tilde{V}_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad \tilde{V}_{d+1} = (0, \dots, 0, 1)$$

and

$$(3.3) \quad \tilde{V}_{d+2} = (-1)^{d+1} \tilde{V}_1.$$

The last vertex has the same projection as the first one;  $S_d$  is contractible for odd  $d$ , and non-contractible for even  $d$ .

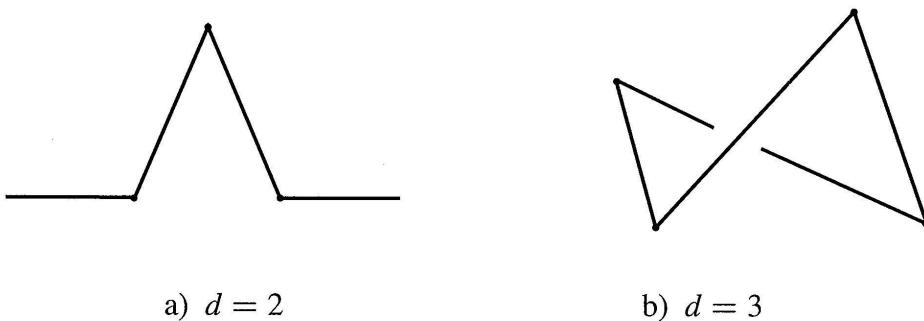


FIGURE 5

PROPOSITION 3.9. *The polygon  $S_d$  is strictly convex.*

*Proof.* We need to prove that through every  $(d-1)$ -tuple

$$(V_1, \dots, \hat{V}_i, \dots, \hat{V}_j, \dots, V_{d+1})$$

there passes a hyperplane  $H$  intersecting  $P$  with multiplicity  $d-1$ . Select a point  $W$  on the line  $(\tilde{V}_i, \tilde{V}_j)$  in such a manner that  $W$  lies on the segment  $(\tilde{V}_i, \tilde{V}_j)$  if  $j-i$  is even, and does not lie on it if  $j-i$  is odd. Define  $\tilde{H}$  as the linear span of  $\tilde{V}_1, \dots, \hat{\tilde{V}}_i, \dots, \hat{\tilde{V}}_j, \dots, \tilde{V}_{d+1}, W$ . We claim that its projection  $H \subset \mathbf{RP}^d$  meets  $S_d$  with multiplicity  $\leq d-1$ .