

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-
VERTEX AND 6-VERTEX THEOREMS
Autor: Ovsienko, V / Tabachnikov, S.
Kapitel: 2.3 DISCRETE GHYS THEOREM
DOI: <https://doi.org/10.5169/seals-65426>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a “dual” theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered n -tuples $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ in \mathbf{RP}^1 with $n \geq 4$. We choose an orientation of \mathbf{RP}^1 and assume that the cyclic ordering of each of the two n -tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in \mathbf{RP}^1 determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers $a < b < c < d$, the cross-ratio is

$$(2.1) \quad [a, b, c, d] = \frac{(c-a)(d-b)}{(b-a)(d-c)}.$$

DEFINITION 2.9. A triple of consecutive indices $(i, i+1, i+2)$ is said to be *extremal* if the difference of cross-ratios

$$(2.2) \quad [y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}]$$

changes sign as j varies from $i-1$ to i (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. For every pair X, Y of n -tuples of points as above, there exist at least four extremal triples.

EXAMPLE 2.11. If $n = 4$ then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$(2.3) \quad [x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1},$$

and this is an involution. Furthermore, if $a > b > 1$ then $a/(a-1) < b/(b-1)$. Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries x_i, x_{i+1}, x_{i+2} into y_i, y_{i+1}, y_{i+2} , respectively. The graph G of this transformation can be seen as a curve in $\mathbf{RP}^1 \times \mathbf{RP}^1$; the three points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ lie

on this graph. An ordered pair of points (x_j, x_{j+1}) in oriented \mathbf{RP}^1 defines a unique segment. An ordered pair of points $((x_j, y_j), (x_{j+1}, y_{j+1}))$ in $\mathbf{RP}^1 \times \mathbf{RP}^1$ also defines a unique segment, namely the one whose projection on each factor is a segment in \mathbf{RP}^1 as defined before. The triple $(i, i+1, i+2)$ is extremal if and only if the topological intersection index of the broken line $(x_{i-1}, y_{i-1}), \dots, (x_{i+3}, y_{i+3})$ with the graph G is zero. This fact can be checked from (2.1) by a direct computation, which we omit.

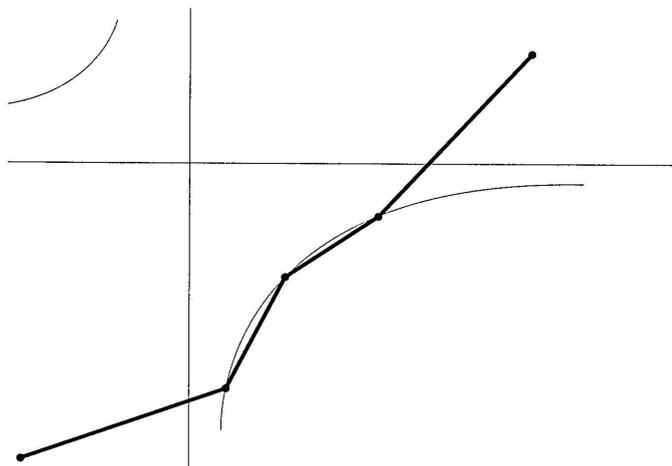


FIGURE 2

Let us also comment on the relation between Definition 2.9 and the zeroes of the Schwarzian derivative of a diffeomorphism of the projective line. Let

$$x_0 = 0, \quad x_1 = \varepsilon, \quad x_2 = 2\varepsilon, \quad x_3 = 3\varepsilon$$

be four infinitely close points given in some affine coordinate, and let $y_i = f(x_i)$ where f is a diffeomorphism of \mathbf{RP}^1 . Then a direct computation using (2.1) yields:

$$[y_0, y_1, y_2, y_3] - [x_0, x_1, x_2, x_3] = \varepsilon^2 S(f)(0) + O(\varepsilon^3),$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of f . Thus, for $\varepsilon \rightarrow 0$, Definition 2.9 corresponds to the vanishing of the Schwarzian derivative.