Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

Autor: Ovsienko, V / Tabachnikov, S.

Kapitel: 2.3 DISCRETE GHYS THEOREM

DOI: https://doi.org/10.5169/seals-65426

REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a "dual" theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered n-tuples $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ in $\mathbf{RP}^1$ with $n \geq 4$. We choose an orientation of $\mathbf{RP}^1$ and assume that the cyclic ordering of each of the two n-tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in $\mathbf{RP}^1$ determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers $a < b < c < d$, the cross-ratio is

$$[a,b,c,d] = \frac{(c-a)(d-b)}{(b-a)(d-c)} . \tag{2.1}$$

DEFINITION 2.9. A triple of consecutive indices $(i, i+1, i+2)$ is said to be *extremal* if the difference of cross-ratios

$$[y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}] \tag{2.2}$$

changes sign as j varies from $i-1$ to i (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. *For every pair X, Y of n-tuples of points as above, there exist at least four extremal triples.*

EXAMPLE 2.11. If $n = 4$ then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$[x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1} , \tag{2.3}$$

and this is an involution. Furthermore, if $a > b > 1$ then $a/(a-1) < b/(b-1)$. Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries x_i, x_{i+1}, x_{i+2} into y_i, y_{i+1}, y_{i+2}, respectively. The graph G of this transformation can be seen as a curve in $\mathbf{RP}^1 \times \mathbf{RP}^1$; the three points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) lie on this graph. An ordered pair of points (x_j, x_{j+1}) in oriented $\mathbf{RP}^1$ defines a unique segment. An ordered pair of points $\big((x_j, y_j), (x_{j+1}, y_{j+1})\big)$ in $\mathbf{RP}^1 \times \mathbf{RP}^1$ also defines a unique segment, namely the one whose projection on each factor is a segment in $\mathbf{RP}^1$ as defined before. The triple $(i, i+1, i+2)$ is extremal if and only if the topological intersection index of the broken line $(x_{i-1}, y_{i-1}), \ldots, (x_{i+3}, y_{i+3})$ with the graph G is zero. This fact can be checked from (2.1) by a direct computation, which we omit.

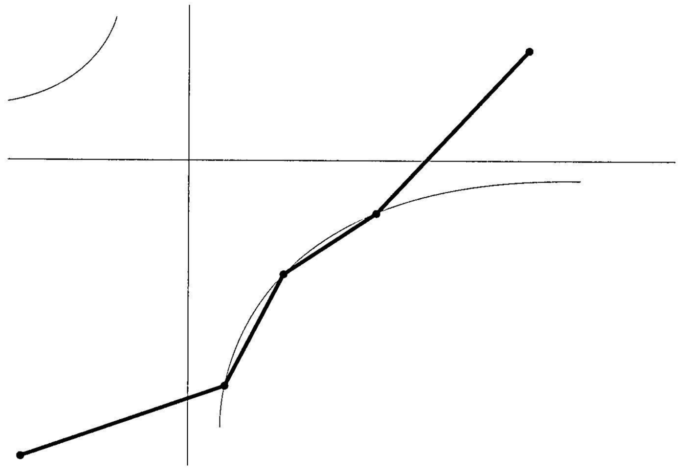

FIGURE 2

Let us also comment on the relation between Definition 2.9 and the zeroes of the Schwarzian derivative of a diffeomorphism of the projective line. Let

$$x_0 = 0, \quad x_1 = \varepsilon, \quad x_2 = 2\varepsilon, \quad x_3 = 3\varepsilon$$

be four infinitely close points given in some affine coordinate, and let $y_i = f(x_i)$ where f is a diffeomorphism of $\mathbf{RP}^1$. Then a direct computation using (2.1) yields:

$$[y_0, y_1, y_2, y_3] - [x_0, x_1, x_2, x_3] = \varepsilon^2 S(f)(0) + O(\varepsilon^3),$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f. Thus, for $\varepsilon \to 0$, Definition 2.9 corresponds to the vanishing of the Schwarzian derivative.