

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	47 (2001)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS
<b>Autor:</b>	Ovsienko, V / Tabachnikov, S.
<b>Kapitel:</b>	2.3 DISCRETE GHYS THEOREM
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-65426">https://doi.org/10.5169/seals-65426</a>

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REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a “dual” theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

### 2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered  $n$ -tuples  $X = (x_1, \dots, x_n)$  and  $Y = (y_1, \dots, y_n)$  in  $\mathbf{RP}^1$  with  $n \geq 4$ . We choose an orientation of  $\mathbf{RP}^1$  and assume that the cyclic ordering of each of the two  $n$ -tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in  $\mathbf{RP}^1$  determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers  $a < b < c < d$ , the cross-ratio is

$$(2.1) \quad [a, b, c, d] = \frac{(c-a)(d-b)}{(b-a)(d-c)}.$$

DEFINITION 2.9. A triple of consecutive indices  $(i, i+1, i+2)$  is said to be *extremal* if the difference of cross-ratios

$$(2.2) \quad [y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}]$$

changes sign as  $j$  varies from  $i-1$  to  $i$  (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. *For every pair  $X, Y$  of  $n$ -tuples of points as above, there exist at least four extremal triples.*

EXAMPLE 2.11. If  $n = 4$  then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$(2.3) \quad [x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1},$$

and this is an involution. Furthermore, if  $a > b > 1$  then  $a/(a-1) < b/(b-1)$ . Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries  $x_i, x_{i+1}, x_{i+2}$  into  $y_i, y_{i+1}, y_{i+2}$ , respectively. The graph  $G$  of this transformation can be seen as a curve in  $\mathbf{RP}^1 \times \mathbf{RP}^1$ ; the three points  $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$  lie

on this graph. An ordered pair of points  $(x_j, x_{j+1})$  in oriented  $\mathbf{RP}^1$  defines a unique segment. An ordered pair of points  $((x_j, y_j), (x_{j+1}, y_{j+1}))$  in  $\mathbf{RP}^1 \times \mathbf{RP}^1$  also defines a unique segment, namely the one whose projection on each factor is a segment in  $\mathbf{RP}^1$  as defined before. The triple  $(i, i+1, i+2)$  is extremal if and only if the topological intersection index of the broken line  $(x_{i-1}, y_{i-1}), \dots, (x_{i+3}, y_{i+3})$  with the graph  $G$  is zero. This fact can be checked from (2.1) by a direct computation, which we omit.

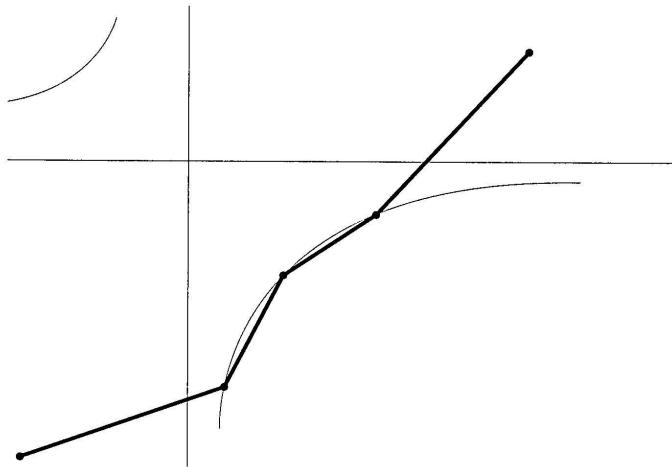


FIGURE 2

Let us also comment on the relation between Definition 2.9 and the zeroes of the Schwarzian derivative of a diffeomorphism of the projective line. Let

$$x_0 = 0, \quad x_1 = \varepsilon, \quad x_2 = 2\varepsilon, \quad x_3 = 3\varepsilon$$

be four infinitely close points given in some affine coordinate, and let  $y_i = f(x_i)$  where  $f$  is a diffeomorphism of  $\mathbf{RP}^1$ . Then a direct computation using (2.1) yields:

$$[y_0, y_1, y_2, y_3] - [x_0, x_1, x_2, x_3] = \varepsilon^2 S(f)(0) + O(\varepsilon^3),$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of  $f$ . Thus, for  $\varepsilon \rightarrow 0$ , Definition 2.9 corresponds to the vanishing of the Schwarzian derivative.