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Artikel: PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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Kapitel: 2.1 DISCRETE 4-VERTEX THEOREM

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2. Theorems on plane polygons

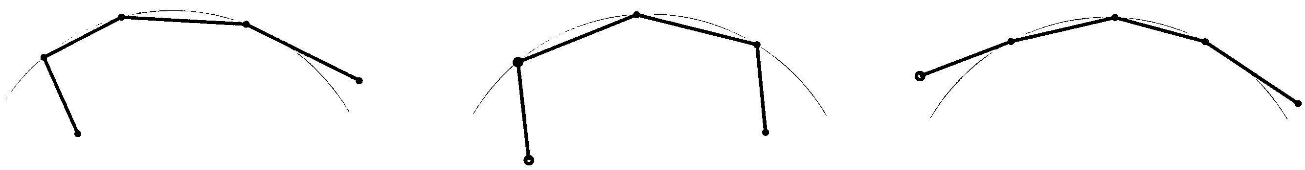
In this section we formulate our results for plane polygonal curves. The proofs will be given in Section 4.1.

2.1 Discrete 4-vertex theorem

The *osculating circle* of a smooth plane curve at a point is the circle (or straight line) that has 3^{rd} order of contact with the curve at the given point. One may say that the osculating circle goes through 3 infinitely close points; at a vertex the osculating circle passes through 4 infinitely close points. Moreover, a generic curve crosses the osculating circle at a generic point and stays on one side of it at a vertex. This well-known fact motivates the following definition.

Let P be a plane convex n-gon; throughout this section we assume that $n \geq 4$. Denote the consecutive vertices by $V_1, \ldots, V_n$; the subscripts are understood cyclically, that is, $V_{n+1} = V_1$, etc.

Definition 2.1. A triple of vertices (V_i, V_{i+1}, V_{i+2}) is said to be *extremal*[1]) if V_{i-1} and V_{i+3} lie on the same side of the circle through V_i, V_{i+1}, V_{i+2} (this does not exclude the case where V_{i-1} or V_{i+3} belongs to the circle).



a) not extremal b) extremal

Figure 1

The next result follows from a somewhat more general theorem due to O. Musin and V. Sedykh [12] (see also [13]).

[1]) We have a terminological difficulty here: as we are dealing with polygons, we cannot use the term "vertex" in the same sense as in the smooth case; hence the term "extremal".

THEOREM 2.2. *Every plane convex polygon P has at least 4 extremal triples of vertices.*

EXAMPLE 2.3. If P is a quadrilateral then the theorem holds tautologically since the $(i-1)^{\mathrm{st}}$ vertex coincides with the $(i+3)^{\mathrm{rd}}$ for every i.

REMARK 2.4. An alternative approach to discretization of the 4-vertex theorem consists in inscribing circles in consecutive triples of sides of a polygon (the centre of such a circle is the intersection point of the bisectors of consecutive angles of the polygon). Then a triple of sides $(\ell_i, \ell_{i+1}, \ell_{i+2})$ is said to be *extremal* if the lines ℓ_{i-1}, ℓ_{i+3} either both intersect the corresponding circle or both fail to intersect it. With this definition an analogue of Theorem 2.2 holds true [19, 16], and this, in the limit, also provides the smooth 4-vertex theorem.

Both formulations, concerning circumscribed or inscribed circles, make sense on the sphere. Moreover, they are equivalent via projective duality.

2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

Five generic points in the plane determine a conic. Considering the plane as an affine part of the projective plane, the complement of the conic has two connected components. Let P be a plane convex n-gon; throughout this section we assume that $n \geq 6$. As in the previous section, we introduce the following definition.

DEFINITION 2.5. Five consecutive vertices $V_i, \ldots, V_{i+4}$ are said to be *extremal* if V_{i-1} and V_{i+5} lie on the same side of the conic through these 5 points (this does not exclude the case where V_{i-1} or V_{i+5} belongs to the conic).

If P is replaced by a smooth convex curve, and $V_i, \ldots, V_{i+4}$ are infinitely close points, we recover the definition of an affine vertex. Hence the following theorem is a discrete version of the smooth theorem on 6 affine vertices.

THEOREM 2.6. *Every plane convex polygon P has at least 6 extremal quintuples of vertices.*

EXAMPLE 2.7. If P is a hexagon then the theorem holds tautologically for the same reason as in Example 2.3.