Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 46 (2000)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ARITHMETIC OF BINARY CUBIC FORMS

Autor: HOFFMAN, J. William / MORALES, Jorge

Kapitel: 7. EXPLICIT COMPUTATIONS AND CUBIC TRACE FORMS

DOI: https://doi.org/10.5169/seals-64795

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 26.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

With this notation we have a commutative square

(32)
$$\begin{tikzcd}
H^1_{\mathrm{fl}}(X,\mu_3) \arrow[r, "i_*"] \arrow[d, "f"'] & \mathrm{Pic}\,(C)[3] \\
\mathcal{S}(C) \arrow[r, "e'"] & H(C)[3] \arrow[u, "j"]
\end{tikzcd}$$

where $j\colon H(C) \to \mathrm{Pic}\,(C)$ is the natural homomorphism $[M,q,N] \mapsto [M]$. Kneser [11, §6] has shown that j is an isomorphism (see also Section 2), so the two vertical maps in (32) are bijections and the horizontal maps are surjections.

Note that because of the exact sequence (31), the fibers of e' are in one-to-one correspondence with the elements of the group $C^\times/C^{\times 3}$. This is, of course, equivalent to Theorem 5.2, Part (ii).

7. EXPLICIT COMPUTATIONS AND CUBIC TRACE FORMS

In this section we assume that $A := C \otimes K$ is a quadratic étale algebra over K. In this case the trace form $(x,y) \to \mathrm{Tr}_{A/K}(xy)$ is nondegenerate and gives rise to a natural isomorphism between the codifferent

$$C' = \{x \in A : \mathrm{Tr}_{A/K}(xC) \subset R\}$$

and the dual C^*. If M is a fractional C-ideal with $M^3 \simeq C'$, then, by Theorem 5.1, the cubic forms on M with primitive determining form are given by

$$F_u(\mathbf{x}) = \mathrm{Tr}_{A/K}(ua\mathbf{x}^3)\,, \tag{33}$$

where $a \in A$ is a fixed element with $aM^3 = C'$, and u is a unit of C. Moreover, by Theorem 5.1, two such forms F_u and F_v are C-isomorphic if and only if u and v represent the same element of $C^\times/(C^\times)^3$.

We shall compute explicitly some examples for $R = \mathbf{Z}$ using (33). In this case we have $C = \mathbf{Z}[t]/(f(t))$, where f is a monic degree-two polynomial with distinct roots and coefficients in $\mathbf{Z}$.

Let ω be the class of t in C. It is well-known, and easy to prove, that the codifferent C' is a principal fractional C-ideal generated by $f'(\omega)^{-1}$, where f' is the derivative of f. Hence, $[C^*]$ is trivial in $\mathrm{Pic}\,(C)$ (note that this holds more generally provided $\mathrm{Pic}\,(R) = 0$).

EXAMPLE 7.1. Let $C = \mathbf{Z}[\frac{1+\sqrt{-23}}{2}]$ (note that 23 is the smallest square-free positive integer N such that $A = \mathbf{Q}(\sqrt{-N})$ has class number divisible by 3; in fact $\mathrm{Pic}\,(C) \simeq \mathbf{Z}/3\mathbf{Z}$ (see [2]). The class group $\mathrm{Pic}\,(C)$ is generated by the class of

$$M = 2\mathbf{Z} + \omega\mathbf{Z},$$

where $\omega = \frac{1+\sqrt{-23}}{2}$. Thus the three classes of $\mathrm{Pic}\,(C)$ are represented by the ideals C, M and $\overline{M}$. The quadratic forms attached to C, M and $\overline{M}$ are respectively

$$x_1^2 + x_1x_2 + 6x_2^2, \qquad 2x_1^2 + x_1x_2 + 3x_2^2, \qquad 2x_1^2 - x_1x_2 + 3x_2^2 .$$

One verifies also that $\theta = \omega - 2$ satisfies $M^3 = \theta C$, thus $(1/\theta\sqrt{-23})M^3 = C'$. Hence, by (33), the cubic C-form on M is given by

$$F(\mathbf{x}) = \mathrm{Tr}\left(\frac{\mathbf{x}^3}{\theta\sqrt{-23}}\right),$$

where $\mathbf{x} = 2x_1 + x_2\omega$. Similar computations can be done for $\overline{M}$ (taking $\theta = -1 - \omega$ and the $\mathbf{Z}$-basis $\{2, -1+\omega\}$) and for C (with the basis $\{1, \omega\}$). The following table summarizes the results of these computations:

Module	Cubic Form	Determining Form
M	$-x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 + 2x_2^3$	$2x_1^2 + x_1x_2 + 3x_2^2$
$\overline{M}$	$x_1^3 - 3x_1^2x_2 - 3x_1x_2^2 + 2x_2^3$	$2x_1^2 - x_1x_2 + 3x_2^2$
C	$x_2(3x_1^2 + 3x_1x_2 - 5x_2^2)$	$x_1^2 + x_1x_2 + 6x_2^2$

EXAMPLE 7.2. Let $C = \mathbf{Z}[\sqrt{79}]$. Here also $\mathrm{Pic}\,(C) \simeq \mathbf{Z}/3\mathbf{Z}$ (see [2]) (in fact 79 is the smallest square-free positive integer N such that $\mathbf{Q}(\sqrt{N})$ has class number divisible by 3).

The class group $\mathrm{Pic}\,(C)$ is generated by the class of

$$M = 9\mathbf{Z} + (4 + \sqrt{79})\mathbf{Z}.$$

Thus the three classes of $\mathrm{Pic}\,(C)$ are represented by the ideals C, M and $\overline{M}$. One verifies also that $\alpha = 52 - 5\sqrt{79}$ satisfies $M^3 = \alpha C$, thus $(1/2\alpha\sqrt{79})M^3 = C'$. The fundamental unit of C is $\tau = 80 + 9\sqrt{79}$; hence, by (33), the three nonisomorphic cubic C-forms on M are given by

$$F_{\tau^k}(\mathbf{x}) = \mathrm{Tr}\left(\frac{\tau^k}{2\alpha\sqrt{79}}\mathbf{x}^3\right),$$

where $\mathbf{x} = 9x_1 + (4+\sqrt{79})x_2$ and $k = -1, 0, 1$. Similar computations can be done for $\overline{M}$ (taking the $\mathbf{Z}$-basis $\{9, -4+\sqrt{79}\}$) and C (with the natural basis $\{1, \sqrt{79}\}$).

Module	Cubic Forms	Determining Form
M	$-68x_1^3 + 111x_1^2x_2 - 60x_1x_2^2 + 11x_2^3$ $5x_1^3 + 24x_1^2x_2 + 33x_1x_2^2 + 16x_2^3$ $868x_1^3 + 3729x_1^2x_2 + 5340x_1x_2^2 + 2549x_2^3$	$9x_1^2 + 8x_1x_2 - 7x_2^2$
$\overline{M}$	$-868x_1^3 + 3729x_1^2x_2 - 5340x_1x_2^2 + 2549x_2^3$ $-5x_1^3 + 24x_1^2x_2 - 33x_1x_2^2 + 16x_2^3$ $68x_1^3 + 111x_1^2x_2 + 60x_1x_2^2 + 11x_2^3$	$9x_1^2 - 8x_1x_2 - 7x_2^2$
C	$-9x_1^3 + 240x_1^2x_2 - 2133x_1x_2^2 + 6320x_2^3$ $3x_1^2x_2 + 79x_2^3$ $9x_1^3 + 240x_1^2x_2 + 2133x_1x_2^2 + 6320x_2^3$	$x_1^2 - 79x_2^2$

REFERENCES

[1] ARNDT, F. Zur Theorie der binären kubischen Formen. *J. Crelle 53* (1857), 309–321.

[2] BOREVICH, Z. I. and I. R. SHAFAREVICH. *Number Theory.* Academic Press, New York, 1966.

[3] CAYLEY, A. Two letters on cubic forms. *Quarterly Math. J. 1* (1857), 85–87, 90–91 = *Collected Mathematical Papers*, vol. 3, 9–12, Cambridge Univ. Press, 1890.

[4] DICKSON, L. E. *History of the Theory of Numbers, vol. III.* Chelsea, 1952.

[5] —— On invariants and the theory of numbers. *The Madison Colloquium.* Dover Publications, New York, 1966.

[6] EISENSTEIN, G. Théorèmes sur les formes cubiques et solution d'une équation du quatrième degré à quatre indéterminées. *J. Crelle 27* (1844), 75–79 = *Mathematische Werke*, Band I, Chelsea Publ. Co., 1975, 1–5.

[7] —— Untersuchungen über die cubischen Formen mit zwei Variabeln. *J. Crelle 27* (1844), 89–104 = Mathematische Werke, Band I, Chelsea Publ. Co., 1975, 10–25.

[8] HERMITE, C. Lettre à Cayley sur les formes cubiques. *Œuvres*, tome 1, 437–439, Gauthier-Villars, 1905.