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LEMMA 4.8. *Suppose that $C \otimes K$ is étale over K and let (M, F) and (M', F') be cubic C -forms. Assume that the determining mappings $q_F, q_{F'}$ are nonzero. Then every R -linear isomorphism $f: (M, F) \rightarrow (M', F')$ is either C -linear or C -sesquilinear.*

Proof. The map f will induce an isomorphism of determining quadratic mappings of type C . We conclude by Proposition 2.3. \square

5. STRUCTURE OF THE CUBIC C -FORMS

We shall describe below the C -module structure of $S_C^3(M^*)$ and the corresponding C -isomorphism classes.

THEOREM 5.1. *Let M be a rank-one projective C -module. For each $\phi \in \text{Hom}_C(M_C^{\otimes 3}, C^*)$ we define a cubic form by $F_\phi(\mathbf{x}) = \phi(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(1)$. Then*

- (i) *The correspondence $\phi \mapsto F_\phi$ is an isomorphism of C -modules $\text{Hom}_C(M_C^{\otimes 3}, C^*) \rightarrow S_C^3(M^*)$.*
- (ii) *The determining mapping q_{F_ϕ} is primitive if and only if ϕ is an isomorphism.*
- (iii) *Two cubic C -forms F and F_1 on M are equivalent over C if and only if there exists $c \in C^\times$ such that $F_1 = c^3 F$.*

Proof. (i) This is a restatement of Proposition 3.7. The map $\phi \mapsto F_\phi$ is a C -isomorphism by definition of the structure of C -module on $S_C^3(M^*)$ in Section 3.

(ii) It is enough to prove our assertion locally, so we assume that M is free over C . Write $M = C\mathbf{m}$ for some $\mathbf{m} \in M$. Let $\lambda = \phi(\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m})$. Then we have $\phi(x\mathbf{m} \otimes y\mathbf{m} \otimes z\mathbf{m}) = \lambda(xyz)$. Let $\beta(y\mathbf{m}, z\mathbf{m}) = \lambda(yz)$ and observe that λ is a basis of C^* over C if and only if the symmetric bilinear form β is unimodular. We have

$$\begin{aligned} q_{F_\phi}(x\mathbf{m}) &= n(x)q_{F_\phi}(\mathbf{m}) \\ &= n(x) \wedge^2 \beta. \end{aligned}$$

It follows from this equality that q_{F_ϕ} is primitive if and only if β is unimodular, that is, if and only if ϕ is an isomorphism.

(iii) Let F and F_1 be cubic C -forms on M . Suppose that they are C -isomorphic. Then there exists $c \in C^\times$ such that $F_1 = F \circ l_c$. Let T be the symmetric trilinear form associated to F . Since $T(cx, cy, cz) = T(c^3x, y, z)$, we get $F_1 = c^3F$. Conversely, if $F_1 = c^3F$ we may reverse these steps to conclude that $F_1 = F \circ l_c$ \square

We shall henceforth denote by $\text{Cubic}_C(M)$ the set of C -isomorphism classes of cubic C -forms on M with primitive determining mapping. Recall that when M is an invertible C -module, there is a *unique* primitive quadratic mapping (M, q, N) of type C on M ([11]). If $F \in \text{Cubic}_C(M)$, then necessarily

$$(M, q_F, \mathcal{D}(M)) = (M, q, N) \text{ in } H(C), \quad \text{and} \quad C = C^+(M, q_F, \mathcal{D}(M)),$$

by Corollary 4.7 (ii); in particular, all members of $\text{Cubic}_C(M)$ have isomorphic determining mappings.

THEOREM 5.2. *Let M be a projective C -module of rank one.*

- (i) *The set $\text{Cubic}_C(M)$ is nonempty if and only if $3[M] = [C^*]$ in $\text{Pic}(C)$.*
- (ii) *If $3[M] = [C^*]$ in $\text{Pic}(C)$, then the group $C^\times / C^{\times 3}$ acts simply transitively on the set $\text{Cubic}_C(M)$.*

Proof. (i) By Part (ii) of Theorem 5.1, the module M admits a cubic C -form with primitive determining mapping if and only if there is an isomorphism $M_C^3 \rightarrow C^*$.

(ii) Since $M_C^{\otimes 3}$ and C^* are invertible C -modules, $\text{Isom}_C(M_C^{\otimes 3}, C^*)$ is either empty or it is a torsor for C^\times (i.e., a simply transitive C^\times -set). It is nonempty if and only if $\text{Cubic}_C(M)$ is nonempty, by Part (i). Suppose this is so, and choose an isomorphism $\phi: M_C^3 \rightarrow C^*$. Each cubic C -form on M with primitive determining mapping is uniquely of the shape $F_{c\phi}$ with $c \in C^\times$ by Parts (i) and (ii) of Theorem 5.1. By Part (iii) of Theorem 5.1, the form $F_{c\phi}$ will be isomorphic with F_ϕ if and only if $c \in (C^\times)^3$. \square

We discuss next the relation between R -isomorphism and C -homomorphism of cubic forms.

Let $\text{Cubic}_R(M)$ be the set of R -isomorphism classes of binary Gaussian cubic forms on M with primitive determining mapping of type C . Set

$$\mathcal{S}_R(C) = \coprod_{[M]} \text{Cubic}_R(M) \quad \text{and} \quad \mathcal{S}(C) = \coprod_{[M]} \text{Cubic}_C(M),$$

where $[M]$ runs over the elements of $\text{Pic}(C)$ satisfying $3[M] = [C^*]$ and \coprod means disjoint union.

The set $\mathcal{S}(C)$ carries a natural involution given by

$$[M, F] \mapsto \overline{[M, F]} := [\overline{M}, F],$$

where \overline{M} is defined as follows: $\overline{M} = M$ as R -modules with C acting by $c \cdot \mathbf{x} = \overline{c}\mathbf{x}$, where $c \mapsto \overline{c}$ is the canonical involution of C . This is well-defined because q_F depends only on the R -module structure of M , and it will be of type C for M if and only if it is so for \overline{M} since $n(c) = n(\overline{c})$. Note that $[M, F] = \overline{[M, F]}$ if and only if (M, F) possesses a C -sesquilinear automorphism.

PROPOSITION 5.3. *With the previous notation we have*

- (i) $\mathcal{S}_R(C) = \mathcal{S}(C) / \sim$, where \sim identifies $[M, F]$ with $\overline{[M, F]}$.
- (ii) If $[M] = [\overline{M}]$ and $3[M] = [C^*]$, then $\text{Cubic}_C(M)$ has an element $[M, F_0]$ fixed under the involution.
- (iii) If $[M] \neq [\overline{M}]$ and $3[M] = [C^*]$ in $\text{Pic}(C)$, then $\text{Cubic}_C(M) = \text{Cubic}_R(M)$. In particular, $\text{Cubic}_R(M)$ is a simply transitive $(C^\times / C^{\times 3})$ -set.

Proof. (i) Let $\psi: (M, F) \rightarrow (M', F')$ be an R -isomorphism. Then ψ is an isomorphism of quadratic mappings $(M, q_F, \mathcal{D}(M)) \rightarrow (M', q_{F'}, \mathcal{D}(M'))$. By Proposition 2.3, the map ψ is either C -linear or C -sesquilinear. Hence either $[M, F] = [M', F']$ or $[M, F] = \overline{[M', F']}$.

(ii) We start out with an element $[M, F] \in \mathcal{S}(C)$, which exists by hypothesis on M and by Theorem 5.2(i), and we choose a C -sesquilinear automorphism $\sigma: M \rightarrow M$. We know by Theorem 5.2 that all the C -forms on M are of the form wF with $w \in C^\times$. In particular $F \circ \sigma = wF$ for some $w \in C^\times$. An easy computation using (21) shows $(wF) \circ \sigma = \overline{w}(F \circ \sigma)$, so $F \circ \sigma^2 = \overline{w}wF$. Since σ^2 is C -linear, it follows from Theorem 5.2 that $\overline{w}w \in C^{\times 3}$. Using the fact that the cohomology of $\mathbf{Z}/2\mathbf{Z}$ with coefficients in a group of odd exponent (in this case $C^\times / C^{\times 3}$ with $\mathbf{Z}/2\mathbf{Z}$ acting via the canonical involution of C) is trivial, we conclude that $w = \overline{u}^{-1}uv^3$ for some $u, v \in C^\times$. Let $F_0 = uF$. By direct computation we have $F_0 \circ \sigma = v^3F_0$; thus $\overline{[M, F]} = [M, F \circ \sigma] = [M, F]$ as claimed.

(iii) If $[M] \neq [\overline{M}]$, by Part (i), no two distinct elements of $\text{Cubic}_C(M)$ can be identified in $\text{Cubic}_R(M)$, that is, the canonical projection

$$\text{Cubic}_C(M) \rightarrow \text{Cubic}_R(M)$$

is a bijection. The second assertion follows from Theorem 5.2. \square

COROLLARY 5.4. *Let $[M] \in \text{Pic}(C)$ be as in Part (ii) of Theorem 5.3. Let $[M, F_0] \in \text{Cubic}_C(M)$ be a the fixed point of the involution. Then the map $(C^\times / C^{\times 3}) \rightarrow \text{Cubic}_C(M)$ given by $u \mapsto [M, uF_0]$ is an isomorphism of $\mathbf{Z}/2\mathbf{Z}$ -sets. In particular, this correspondence induces a bijection $\text{Cubic}_R(M) \simeq (C^\times / C^{\times 3}) / \sim$, where \sim identifies c with \bar{c} .*

Proof. By Theorem 5.2, it is enough to show that the map $u \mapsto [M, uF_0]$ commutes with the action of $\mathbf{Z}/2\mathbf{Z}$ via the involutions. Let $\sigma: (\bar{M}, F_0) \rightarrow (M, F_0)$ be a C -isomorphism and let $u \in C^\times$. Since $(uF_0) \circ \sigma = \bar{u}(F_0 \circ \sigma)$, we have $[\bar{M}, uF_0] = [\bar{M}, uF_0] \stackrel{\sigma}{=} [M, (uF_0) \circ \sigma] = [M, \bar{u}(F_0 \circ \sigma)] = [M, \bar{u}F_0]$. \square

The above proposition applies in particular to the case of fields. We can summarize our results in this case as follows:

PROPOSITION 5.5. *Let K be a field of characteristic not 2 or 3. Let \mathcal{S}_K be the set of K -isomorphism classes of all binary cubic forms over K with nonzero discriminant. Then there is a natural partition*

$$(25) \quad \mathcal{S}_K = \coprod_C \text{Cubic}_K(C),$$

where C ranges over the quadratic étale K -algebras and each $\text{Cubic}_K(C)$ is in one-to-one correspondence with the quotient of $C^\times / (C^\times)^3$ by the involution $c \mapsto \bar{c}$.

Proof. If K is a field then $\text{Pic}(C) = 0$ for all quadratic K -algebras C . Each cubic form with nonzero discriminant will be a C -form for a unique quadratic étale algebra, namely the even Clifford algebra of its determining form, by Proposition 2.8 and Theorem 4.5. We finish by applying Proposition 5.3. \square

As an illustration of these ideas, we prove a result known to L. E. Dickson [5, page 23]:

PROPOSITION 5.6. *Let $K = \mathbf{F}_q$ be a finite field with q elements, not of characteristic 2 or 3. Then the number of $\text{GL}_2(\mathbf{F}_q)$ -equivalence classes of binary cubic forms over \mathbf{F}_q with nonzero discriminant is 3 if $q \equiv 2 \pmod{3}$, and is 9 if $q \equiv 1 \pmod{3}$.*

Proof. The étale quadratic algebras over \mathbf{F}_q are

1. $C = \mathbf{F}_q \times \mathbf{F}_q$;
2. $C = \mathbf{F}_{q^2}$.

If $q \equiv 2 \pmod{3}$, then $C^\times / (C^\times)^3$ is trivial in the first case and is $\mathbf{Z}/3\mathbf{Z}$ in the second case since $q^2 \equiv 1 \pmod{3}$. In the second case the involution $c \rightarrow \bar{c}$ fixes the identity element of $C^\times / (C^\times)^3$ and interchanges the other two elements, giving 2 orbits on this. This gives $1 + 2$ orbits in total, so by Proposition 5.5, we have 3 isomorphism classes of binary cubic forms. If $q \equiv 1 \pmod{3}$, then $C^\times / (C^\times)^3$ is $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ in the first case and is $\mathbf{Z}/3$ in the second case. In the second case, the Galois involution acts trivially, since $\mathbf{F}_q^\times / (\mathbf{F}_q^\times)^3 = C^\times / (C^\times)^3$. This gives 3 orbits. In the first case, the involution flips the two factors, and there are clearly 6 orbits. This gives a total of 9 orbits, and hence 9 cubic forms. \square

REMARK 5.7. When $R = K$ is a field of characteristic not 2 or 3, one can give an alternate description of \mathcal{S}_R . Since \mathbf{GL}_2 acts threefold transitively on \mathbf{P}^1 , any binary cubic form with nonzero discriminant is equivalent over the separable closure of K with $\Phi = xy(x - y)$. Therefore, by the usual descent yoga, there is a canonical bijection

$$(26) \quad \mathcal{S}_K \simeq H^1(K, \text{Aut}(\Phi)),$$

where $\text{Aut}(\Phi)$ is the K -group scheme of automorphisms of Φ . The structure of $\text{Aut}(\Phi)$ is easily worked out:

$$\text{Aut}(\Phi) = \mu_3 \times S_3,$$

where S_3 is the symmetric group on 3 letters as a trivial Galois module; it corresponds to the stabilizer in \mathbf{PGL}_2 of the set of zeros of Φ in \mathbf{P}^1 .

The signature $S_3 \rightarrow \mu_2$ induces a homomorphism $\delta: \text{Aut}(\Phi) \rightarrow \mu_2$, which in turn induces a map in Galois cohomology

$$(27) \quad \delta_*: H^1(K, \text{Aut}(\Phi)) \rightarrow H^1(K, \mu_2) = K^\times / K^{\times 2}.$$

Using (4) and the identification (26), we can show that

$$\delta_*(F) = -3D_F \in K^\times / K^{\times 2}.$$

Thus we can interpret the partition (25) as the partition on $H^1(K, \text{Aut}(\Phi))$ given by the fibers of δ_* , the set $\text{Cubic}_K(C)$ corresponding to the fiber $\delta_*^{-1}(-3D)$, where D is the discriminant of C .

When R is a PID we can give a more precise version of Theorem 5.2. In this case, C is a free R -module, and since $R1$ is a direct factor, $C = R \oplus R\omega = R[\omega]$ is a monogenic R -algebra. Therefore C^* is free of rank one over C (see Section 7), so the condition $3[M] = [C^*]$ of Theorem 5.2 reads simply $3[M] = 0$. Furthermore, since $\text{Pic}(R) = 0$, the exact sequence (13) induces an isomorphism

$$(28) \quad G(C)[3] \simeq H(C)[3] = \text{Pic}(C)[3]$$

(note that $R^\times/n(C^\times)$ is an elementary abelian 2-group).

The isomorphism (28) suggests that when R is a PID, it should be possible to use quadratic forms instead of quadratic mappings and develop a theory for binary cubic forms that is completely parallel to Eisenstein's theory over \mathbf{Z} . As we mentioned above, any projective R -module is free, so that a quadratic form (M, q) is the same thing as a quadratic form classically understood: a homogeneous polynomial of degree two. If q is of type C then $M = R^2$ becomes an invertible C -module. This C -module is said to be *associated to* q .

We begin by proving an easy technical lemma.

LEMMA 5.8. *Suppose that R is a UFD and let $C = R[t]/(t^2 + bt + c)$. Let $D = b^2 - 4c$ and let ω be the class of t in C . Set $\delta = b + 2\omega$ (note that $\delta^2 = D$) and let $\xi = x + y\delta$ with $x, y \in R$. If $n(\xi) \equiv 0 \pmod{4R}$, then $\xi \equiv 0 \pmod{2C}$.*

Proof. It is enough to prove $x \equiv by \pmod{2R}$. Let $p \in R$ be an irreducible element. For $z \in R - \{0\}$ we denote by $\text{ord}_p(z)$ the largest power of p occurring in the factorization of z . Set $m = \text{ord}_p(x - by)$. If $m < \text{ord}_p(2)$ then, since ord_p is a valuation, $\text{ord}_p(x + by) = \text{ord}_p(x - by + 2by) = m$. Hence $\text{ord}_p(x^2 - b^2y^2) = 2m < \text{ord}_p(4)$, which contradicts our assumption (since $b^2 \equiv D \pmod{4R}$). Therefore $\text{ord}_p(x - by) \geq \text{ord}_p(2)$ for all irreducible p , which proves the lemma. \square

Now we can prove:

PROPOSITION 5.9. *Let R be a PID and let F be a cubic form on $M = R^2$ given in the natural basis by (1), with coefficients $a_i \in R$. Suppose that its Eisenstein determining form $q_F(\mathbf{x}) = ax_1^2 + bx_1x_2 + cx_2^2$, as in (2), is primitive of discriminant $D \neq 0$ and let $C := C^+(q_F) = R[t]/(t^2 + bt + ac)$. Then $3[M, q_F] = 0$ in $G(C)$.*

Proof. By the syzygy (7) we have

$$4q_F(\mathbf{x})q_F(\mathbf{y})q_F(\mathbf{z}) = X^2 - DY^2,$$

where X and Y are symmetric R -trilinear forms in $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Applying the lemma to the rings $R' := R[x_1, x_2, y_1, y_2, z_1, z_2]$ and $C' := C \otimes_R R'$ with $\xi = X + \delta Y$ (with δ as in the lemma; the lemma applies since R , hence R' , is a UFD), we have

$$(29) \quad q_F(\mathbf{x})q_F(\mathbf{y})q_F(\mathbf{z}) = n(T),$$

where $T = \xi/2 \in C'$, by the lemma. Note that T is symmetric trilinear in $\mathbf{x}, \mathbf{y}, \mathbf{z}$; hence the identity (29) shows that the triplication of q_F is the trivial form, as desired. \square

The results below were essentially known in the case $R = \mathbf{Z}$ to Eisenstein [6] and [7], Arndt [1], Pepin [13], Cayley [3] and Hermite [8].

THEOREM 5.10. *Let R be a PID. Let $q = ax_1^2 + bx_1x_2 + cx_2^2$ be a primitive binary quadratic form over R of discriminant $D = b^2 - 4ac \neq 0$. Let $C = C^+(q)$ be the even Clifford algebra of q and let $M := R^2$ be endowed with the natural C -module structure. Let $\tau \in C$ be such that $\tau + \bar{\tau} = 0$ and $\tau^2 = D$. With this notation we have*

- (i) *There exists a Gaussian binary cubic form F such that $q_F = q$ (where q_F is given by (2)) if and only if $3[M, q] = 0$ in the group $G(C)$ of C -isomorphism classes of quadratic forms of type C .*
- (ii) *If F and F' are Gaussian binary cubic forms with $q_F = q_{F'} = q$, then there exists a unit $c = a + b\tau \in C^\times$ with $n(c) = 1$ such that $F' = cF = aF + bG_F$, where G_F is the cubic covariant defined in (5).*
- (iii) *Let two cubic forms F and F' with $q_F = q_{F'} = q$ be given. The following conditions are equivalent:*
 - (a) *There exists $d \in C^\times$ with $n(d) = 1$ such that $F' = d^3F$.*
 - (b) *There exists $d \in C^\times$ such that $F' = d^3F$.*
 - (c) *F and F' are $\mathbf{SL}_2(R)$ -equivalent.*

Proof. (i) By Proposition 5.9 the condition $3[M, q] = 0$ is necessary. We shall see that it is sufficient. Suppose $3[M, q] = 0$ in $G(C)$; in particular

$$3[M] = 0 \in \text{Pic}(C),$$

thus by virtue of Theorem 5.2, Part (i), there exists a Gaussian cubic form F such that $[M, q_F, R] = [M, q, R]$ in $H(C)$. By Proposition 5.9, the class $[M, q_F]$ is in $G(C)[3]$; hence, by the isomorphism (28), we conclude $[M, q_F] = [M, q]$ in $G(C)$.

(ii) Suppose that $q_F = q_{F'} = q$. $C \otimes K$ is an étale K -algebra since $D \neq 0$. Hence by Corollary 4.7 both F and F' are C -forms and by Theorem 5.2, Part (ii), there exists $c \in C^\times$ such that $F' = cF = (\rho(c)/3)F$ (in the notation of (23)). Writing $c = a + b\tau$ we get $F' = aF + (b/3)(\rho(\tau)F)$. By (24) we have $\rho(\tau)F = 3G_F$ (changing the sign of τ if needed) and direct computation shows $q_{F'} = n(c)q_F$. Thus $n(c) = 1$ as required. Note that in general, the coefficients a, b will have a 2 in the denominator since $\tau = b + 2\omega$ for a generator ω of the algebra C (see Lemma 5.8).

(iii) a) \Rightarrow b) is trivial.

b) \Rightarrow c). If $F' = d^3F$ with $d \in C^\times$ then, by Part (ii) of Theorem 5.2, F and F' are C -equivalent, the isomorphism being $\mathbf{x} \rightarrow d\mathbf{x}$. We have $n(d)^3 = 1$ by the proof of Part (ii) of this theorem, so replacing d by $n(d)d$ we can assume $n(d) = 1$; that is, F and F' are $\mathbf{SL}_2(R)$ -equivalent, and this also establishes the implication b) \Rightarrow a).

c) \Rightarrow a). If $F'(\mathbf{x}) = F(d\mathbf{x})$, where $d \in \mathbf{SL}_2(R)$, then d is in the orthogonal group of $q = q_F = q_{F'}$. Since $\det(d) = 1$, it is in the special orthogonal group of this form, hence given by multiplication by an element $d \in C_1^\times$ by Corollary 2.4. But $F(d\mathbf{x}) = (d^3F)(\mathbf{x})$. \square

COROLLARY 5.11. *Now let $R = \mathbf{Z}$, and let D be a nonzero integer congruent to 0 or 1 modulo 4. Let F be an integral Gaussian binary cubic form with primitive determining form of discriminant D .*

- (i) *Suppose $D < -3$. If F' is another Gaussian binary cubic form with $q_{F'} = q_F$ then F' is $\mathbf{SL}_2(\mathbf{Z})$ -equivalent to F .*
- (ii) *Suppose $D > 0$ or $D = -3$. Then there are exactly three $\mathbf{SL}_2(\mathbf{Z})$ -equivalence classes of Gaussian binary cubic forms F' such that $q_{F'} = q_F$.*

Proof. We have that $C^+(q_F) = C_D$, the unique quadratic \mathbf{Z} -algebra of discriminant D . Note that $(C_D)_1^\times / (C_D)_1^{\times 3}$ is trivial when $D < -3$ and is cyclic of order 3 when $D = -3$ or $D > 0$. The corollary follows immediately from this and Parts (ii) and (iii) of Theorem 5.10. \square

COROLLARY 5.12. *Let D be a nonzero integer congruent to 0 or 1 modulo 4. Let $h_3(D)$ be the number of $\mathbf{SL}_2(\mathbf{Z})$ -equivalence classes of binary Gaussian cubic forms with primitive determining form of discriminant D . Then $h_3(D) = |\text{Pic}(C_D)[3]|$ if $D < -3$ and $h_3(D) = 3|\text{Pic}(C_D)[3]|$ if $D = -3$ or $D > 0$.*

Proof. Follows immediately from Corollary 5.11, equation (28) and Part (i) of Theorem 5.10. \square

6. COHOMOLOGICAL INTERPRETATION

Let \mathbf{G}_m be the multiplicative group regarded as an affine group scheme over $X := \text{Spec } C$ and let $\mu_3 \subset \mathbf{G}_m$ be the kernel of multiplication by 3. All the cohomology groups below are with respect to the flat topology on X .

THEOREM 6.1. *Suppose $[C^*]$ is divisible by 3 in $\text{Pic}(C)$. Then the group $H_{\text{fl}}^1(X, \mu_3)$ acts simply transitively on the set $\mathcal{S}(C)$ of C -equivalence classes of cubic C -forms with primitive determining mapping.*

Proof. Recall that the group $H_{\text{fl}}^1(X, \mu_3)$ can be interpreted concretely as the set of isomorphism classes of pairs (L, ψ) , where L is an invertible C -module and where $\psi: L_C^{\otimes 3} \rightarrow C$ is an isomorphism (see Milne [14, Chap. III, §4]). Let $[L, \psi]$ be an element of $H_{\text{fl}}^1(X, \mu_3)$ and let (M, F) be a cubic C -form. By Theorem 5.1, Part (i), we can assume $F = F_\phi$, where $\phi: M^{\otimes 3} \rightarrow C^*$ is an isomorphism. We define an action of $H_{\text{fl}}^1(X, \mu_3)$ on $\mathcal{S}(C)$ by

$$(30) \quad [L, \psi] \cdot [M, F_\phi] = [L \otimes M, F_{\psi \otimes \phi}],$$

noting that

$$(L \otimes M)_C^{\otimes 3} = L_C^{\otimes 3} \otimes M_C^{\otimes 3} \xrightarrow{\psi \otimes \phi} C \otimes C^* = C^*$$

is an isomorphism. Let us show first that this action is simple. Suppose $[L \otimes M, F_{\psi \otimes \phi}] = [M, F_\phi]$. Then, $L \cong C$. Choosing an isomorphism $L \rightarrow C$, we have $\psi(x \otimes y \otimes z) = uxyz$, where $u \in C^\times$. Hence $[M, F_\phi] = [M, F_{u\phi}]$, and by Part (iii) of Theorem 5.1 we conclude that $u = c^3$ for some $c \in C^\times$. But then $c: C \rightarrow C$ provides an isomorphism of (C, ψ) with $(C, 1)$, thus $[L, \psi] = [C, 1]$.

We show now that the action is transitive. Let $[M_i, F_{\phi_i}]$ ($i = 1, 2$) be elements of $\mathcal{S}(C)$. Let $M_2^\bullet = \text{Hom}_C(M_2, C)$ and let $\phi_2^\bullet: (C^*)^\bullet \rightarrow (M_2^{\otimes 3})^\bullet$ be the dual of ϕ_2 . Let $L = M_1 \otimes M_2^\bullet$ and let $\psi = \phi_1 \otimes \phi_2^{\bullet -1}$. One verifies immediately that $[L, \psi] \cdot [M_2, F_{\phi_2}] = [M_1, F_{\phi_1}]$, which proves that the action is transitive. \square