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## 4. A LIE ALGEBRA REPRESENTATION

Let  $M$  be a projective  $R$ -module of rank two. Let  $G = \text{Aut}_R(M)$  and let  $\mathfrak{g} = \text{End}_R(M)$  viewed as a Lie algebra over  $R$ .

The group  $G$  acts on the right on  $\text{Sym}_R(M^*)$  by algebra automorphisms via

$$(F\sigma)(\mathbf{x}) = F(\sigma\mathbf{x})$$

for  $F \in \text{Sym}_R(M^*)$  and  $\sigma \in G$ . Taking the formal derivative at the origin of the associated map

$$G \rightarrow \text{Aut}_{R\text{-alg}}(\text{Sym}_R(M^*))$$

we get a representation of Lie algebras

$$(22) \quad \rho: \mathfrak{g} \longrightarrow \text{Der}_R(\text{Sym}_R(M^*)).$$

The action of  $G$  preserves the homogeneous components  $\text{Sym}_R^n(M^*)$  and also the submodule  $S^n(M^*)$  of Gaussian forms. The same is true for the Lie algebra action of  $\mathfrak{g}$ .

We shall compute the action of  $\mathfrak{g}$  on  $S^n(M^*)$  explicitly:

LEMMA 4.1. *Let  $F \in S^n(M^*)$  and let  $T$  be the associated  $n$ -linear form. Then*

$$\rho(g)(F)(\mathbf{x}) = nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$$

for all  $g \in \mathfrak{g}$ .

*Proof.* To compute the derivative of  $G \rightarrow \text{Aut}_R(S^n(M^*))$ , we extend the scalars to the “dual numbers”  $R[\epsilon]/(\epsilon^2)$ . Using the symmetry of  $T$  we have

$$F((1 + g\epsilon)\mathbf{x}) = F(\mathbf{x}) + nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})\epsilon,$$

which proves our assertion.  $\square$

Let  $C/R$  be a quadratic algebra in the sense of Section 2 and let  $M$  be an invertible  $C$ -module. Then we have a natural map  $C \rightarrow \text{End}_R(M)$  and we can restrict the representation  $\rho$  to  $C$ . Note that when  $R$  is a field and  $C$  is an étale quadratic algebra then the image of  $C$  is a Cartan subalgebra  $\mathfrak{h}_C$  of  $\mathfrak{g}$ .

Comparing (22) with equation (21), we see that the  $C$ -module structure on  $S_C^3(M^*)$  is related to the Lie algebra action by

$$(23) \quad cF = \frac{1}{3}\rho(c)F.$$

We will make this explicit in a special case that we need:

LEMMA 4.2. Let  $F \in S^3(M^*)$  be a binary cubic form over a field  $K$  of characteristic not 2 or 3. Let  $q_F$  be its determining form, and  $C = C^+(q_F)$  its even Clifford algebra. Let  $x_1, x_2$  be coordinates on the vector space  $M$  with respect to a basis  $\mathbf{m}_1, \mathbf{m}_2$ . Let

$$\tau = \mathbf{m}_1 \mathbf{m}_2 - \mathbf{m}_2 \mathbf{m}_1 \in C = C^+(q_F).$$

Note that  $\tau^2 = D$  is the discriminant of  $q_F$ . Then

$$\rho(\tau) = \frac{\partial q_F}{\partial x_2} \frac{\partial}{\partial x_1} - \frac{\partial q_F}{\partial x_1} \frac{\partial}{\partial x_2},$$

acting on forms of any degree.

*Proof.* As we have seen,

$$q_F(x_1 \mathbf{m}_1 + x_2 \mathbf{m}_2) = Px_1^2 + Qx_1x_2 + Rx_2^2,$$

where  $P = a_1^2 - a_0a_2$ ,  $Q = a_1a_2 - a_0a_3$ , and  $R = a_2^2 - a_1a_3$ . By direct computation in the Clifford algebra  $C$ , we see that

$$\tau \mathbf{m}_1 = Q\mathbf{m}_1 - 2P\mathbf{m}_2$$

$$\tau \mathbf{m}_2 = 2R\mathbf{m}_1 - Q\mathbf{m}_2.$$

Since  $\rho(c)$  is a derivation of  $\text{Sym}_R(M^*)$ , we have

$$\rho(c) = \rho(c)(x_1) \frac{\partial}{\partial x_1} + \rho(c)(x_2) \frac{\partial}{\partial x_2}.$$

Thus  $\tau(x_1 \mathbf{m}_1 + x_2 \mathbf{m}_2) = (Qx_1 + 2Rx_2)\mathbf{m}_1 - (2Px_1 + Qx_2)\mathbf{m}_2$ , which gives  $\rho(\tau)(x_1) = \partial q_F / \partial x_2$  and  $\rho(\tau)(x_2) = -\partial q_F / \partial x_1$ .  $\square$

COROLLARY 4.3.  $\rho(\tau)q_F = 0$  and

$$(24) \quad \begin{aligned} \rho(\tau)F &= \begin{vmatrix} \partial F / \partial x_1 & \partial F / \partial x_2 \\ \partial q_F / \partial x_1 & \partial q_F / \partial x_2 \end{vmatrix} \\ &= 3G_F, \end{aligned}$$

where  $G_F$  is as in (5).

REMARK 4.4. If we further assume that  $C$  is an étale algebra, then as we have remarked,  $\rho$  maps  $C$  onto a Cartan subalgebra of  $\text{End}_K(M) \sim \mathfrak{gl}(2, K)$ . This algebra decomposes as

$$\mathfrak{h}_C = \mathfrak{z} \oplus \mathfrak{h}'_C$$

where the first factor is the center, consisting of scalar matrices, and the second factor is the intersection  $\mathfrak{h}_C \cap \mathfrak{sl}(2, K)$ , consisting of matrices of trace 0. As the formulas in the proof of the preceding lemma show that  $\tau$  acts on  $M$  with trace 0, we see that  $\mathfrak{h}'_C = K\tau$ .

**THEOREM 4.5.** *Let  $C/R$  be a quadratic algebra such that  $C \otimes K$  is étale over  $K$ . Let  $M$  be a projective rank-one  $C$ -module and let  $F \in S^3(M^*)$  be such that the determining mapping  $q_F$  is not 0. Then the following conditions are equivalent:*

- (a)  $F$  is a  $C$ -form
- (b)  $(M, q_F, \mathcal{D}(M))$  is of type  $C$
- (c)  $\rho(c)\rho(\bar{c})F = 9n(c)F$  for all  $c \in C$ .

*Proof.* (a) $\Rightarrow$ (b). If  $T$  is the trilinear form attached to  $F$ , then, using the symmetry of  $T(c\mathbf{x}, \mathbf{y}, \mathbf{z})$ , we have

$$\begin{aligned} q_F(c\mathbf{x}) &= \wedge^2 T(c\mathbf{x}, -, -) \\ &= \wedge^2 (T(\mathbf{x}, c-, -)) \\ &= n(c) \wedge^2 (T(\mathbf{x}, -, -)) \\ &= n(c)q_F(\mathbf{x}), \end{aligned}$$

which proves the claim. In fact, this implication does not depend on  $C \otimes K$  being étale.

It is enough to prove the theorem for the case where  $R = K$  is a separably closed field. We can assume in this case  $C = K[\sigma]$  with  $\sigma$  satisfying  $\sigma^2 = 1$ . We will make these assumptions for the rest of the proof.

(b) $\Rightarrow$ (c). Let  $\{\mathbf{m}_1, \mathbf{m}_2\}$  be a basis of  $M$  over  $K$  with  $\sigma\mathbf{m}_1 = \mathbf{m}_1$  and  $\sigma\mathbf{m}_2 = -\mathbf{m}_2$ . With respect to this basis, the form  $q_F$ , being of type  $C$ , must have the shape

$$q_F(\mathbf{x}) = \alpha x_1 x_2,$$

where  $\alpha \neq 0$ . To see that this is so, note that because  $q_F$  is of type  $C$ , we have  $q_F(\sigma\mathbf{m}_1) = n(\sigma)q_F(\mathbf{m}_1) = -q_F(\mathbf{m}_1)$ , which shows that  $q_F(\mathbf{m}_1) = 0$ . One sees similarly that  $q_F(\mathbf{m}_2) = 0$ . Then the coefficients of  $F(\mathbf{x}) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3$  satisfy the relations:  $a_1^2 - a_0 a_2 = 0$ ,  $a_1 a_2 - a_0 a_3 = \alpha$  and  $a_2^2 - a_1 a_3 = 0$ . Since  $\alpha \neq 0$ , it follows at once that  $a_1 = a_2 = 0$ , so  $F$  is of the form  $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$ . Since  $q_F \neq 0$  (in fact nondegenerate under the étaleness hypothesis), the algebra  $C$  can be identified with the even Clifford algebra  $C^+(M, q_F, \mathcal{D}(M))$  by Proposition 2.8. Under that identification we have  $\tau = \alpha\sigma$ , where  $\tau$  is defined as in Lemma 4.2. From that lemma we get  $\rho(\sigma) = x_1 \partial / \partial x_1 - x_2 \partial / \partial x_2$ , which can be seen directly, since both sides agree on  $x_1, x_2$ . Hence  $\rho(\sigma)(x_1^{3-i} x_2^i) = (3 - 2i)x_1^{3-i} x_2^i$ . In particular, for  $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$  we have

$$\rho(\sigma)\rho(\bar{\sigma})F = -\rho(\sigma)^2 F = -9F = 9n(\sigma)F.$$

The more general identity  $\rho(c)\rho(\bar{c})F = 9n(c)F$  for  $c \in C$  follows from this particular case by noting that, from Lemma 4.1,  $\rho(1)F = 3F$ .

(c) $\Rightarrow$ (a). Suppose that  $\rho(\sigma)^2F = 9F$ . Then  $F$  must have the form  $F = \lambda x_1^3 + \mu x_2^3$ . This is because, as we saw in the discussion above, the monomials  $x_1^{3-i}x_2^i$  are eigenvectors for the operator  $\rho(\sigma)^2$  with eigenvalue  $(3-2i)^2$ , which equals 9 only for  $i = 0$  and  $i = 3$ . Hence the associated trilinear form is  $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 + \mu x_2 y_2 z_2$ . Thus  $T(\sigma \mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 - \mu x_2 y_2 z_2$ , which is visibly symmetric in  $\mathbf{x}, \mathbf{y}, \mathbf{z}$ .  $\square$

REMARK 4.6. It is interesting to notice that the syzygy (6) can be recovered from Part (c) of Theorem 4.5. Assume for simplicity that  $R = K$  is a field and  $C$  is an étale  $K$ -algebra. Let  $\{\mathbf{m}_1, \mathbf{m}_2\}$  be a basis of  $M$ . Let  $\tau = \mathbf{m}_1 \mathbf{m}_2 - \mathbf{m}_2 \mathbf{m}_1 \in C = C^+(q_F)$  as in Lemma 4.2. As we noted in Remark 4.4,  $\tau$  generates the trace 0 part of the Cartan subalgebra defined by  $C$ . Using the derivation property and Corollary 4.3, we see  $\rho(\tau)(G_F^2 - DF^2) = (2/3)(\rho(\tau)^2F - 9DF)G_F$ . From the above theorem,  $\rho(\tau)^2F = 9DF$ , so this is 0. On the other hand,  $\rho(\tau)q_F = 0$ , also by Corollary 4.3, which implies that  $\rho(\tau)q_F^3 = 0$ . Hence both  $q_F^3$  and  $G_F^2 - DF^2$  lie in the subspace on weight 0 (for the action of the Cartan subalgebra  $\mathfrak{h}'_C \subset \mathfrak{sl}(2, K)$ ) of  $S^6(M^*)$ . As  $S^6(M^*)$  is an irreducible representation of  $\mathfrak{sl}(2, K)$ , this is one-dimensional. Hence  $q_F^3$  and  $G_F^2 - DF^2$  differ by a constant multiple. A priori, this constant could depend on  $F$  (e.g.,  $D$ ). That this is not so can be seen by noting that both sides are of the same degree in the coefficients of  $F$ .

COROLLARY 4.7. *Let  $M$  be a projective  $R$ -module of rank 2, and let  $F \in S^3(M^*)$ .*

- (i) *Let  $C = C^+(M, q_F, \mathcal{D}(M))$  and suppose that  $C \otimes K$  is étale, and that  $q_F$  is primitive. Then  $F$  is a  $C$ -form.*
- (ii) *If  $F$  is a  $C$ -form for a quadratic  $R$ -algebra  $C$  and  $(M, q_F, \mathcal{D}(M))$  is primitive, then  $C \cong C^+(M, q_F, \mathcal{D}(M))$ .*

*Proof.* (i) By Proposition 2.8,  $(M, q_F, \mathcal{D}(M))$  is of type  $C$ . We conclude by Theorem 4.5.

(ii) If  $F$  is a  $C$ -form, then by Theorem 4.5, the quadratic mapping  $(M, q_F, \mathcal{D}(M))$  is type  $C$ . But assuming furthermore that  $(M, q_F, \mathcal{D}(M))$  is primitive, we see that  $C$  is isomorphic with  $C^+(M, q_F, \mathcal{D}(M))$  by Proposition 2.8.  $\square$

LEMMA 4.8. *Suppose that  $C \otimes K$  is étale over  $K$  and let  $(M, F)$  and  $(M', F')$  be cubic  $C$ -forms. Assume that the determining mappings  $q_F, q_{F'}$  are nonzero. Then every  $R$ -linear isomorphism  $f: (M, F) \rightarrow (M', F')$  is either  $C$ -linear or  $C$ -sesquilinear.*

*Proof.* The map  $f$  will induce an isomorphism of determining quadratic mappings of type  $C$ . We conclude by Proposition 2.3.  $\square$

## 5. STRUCTURE OF THE CUBIC $C$ -FORMS

We shall describe below the  $C$ -module structure of  $S_C^3(M^*)$  and the corresponding  $C$ -isomorphism classes.

THEOREM 5.1. *Let  $M$  be a rank-one projective  $C$ -module. For each  $\phi \in \text{Hom}_C(M_C^{\otimes 3}, C^*)$  we define a cubic form by  $F_\phi(\mathbf{x}) = \phi(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(1)$ . Then*

- (i) *The correspondence  $\phi \mapsto F_\phi$  is an isomorphism of  $C$ -modules  $\text{Hom}_C(M_C^{\otimes 3}, C^*) \rightarrow S_C^3(M^*)$ .*
- (ii) *The determining mapping  $q_{F_\phi}$  is primitive if and only if  $\phi$  is an isomorphism.*
- (iii) *Two cubic  $C$ -forms  $F$  and  $F_1$  on  $M$  are equivalent over  $C$  if and only if there exists  $c \in C^\times$  such that  $F_1 = c^3 F$ .*

*Proof.* (i) This is a restatement of Proposition 3.7. The map  $\phi \mapsto F_\phi$  is a  $C$ -isomorphism by definition of the structure of  $C$ -module on  $S_C^3(M^*)$  in Section 3.

(ii) It is enough to prove our assertion locally, so we assume that  $M$  is free over  $C$ . Write  $M = C\mathbf{m}$  for some  $\mathbf{m} \in M$ . Let  $\lambda = \phi(\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m})$ . Then we have  $\phi(x\mathbf{m} \otimes y\mathbf{m} \otimes z\mathbf{m}) = \lambda(xyz)$ . Let  $\beta(y\mathbf{m}, z\mathbf{m}) = \lambda(yz)$  and observe that  $\lambda$  is a basis of  $C^*$  over  $C$  if and only if the symmetric bilinear form  $\beta$  is unimodular. We have

$$\begin{aligned} q_{F_\phi}(x\mathbf{m}) &= n(x)q_{F_\phi}(\mathbf{m}) \\ &= n(x) \wedge^2 \beta. \end{aligned}$$

It follows from this equality that  $q_{F_\phi}$  is primitive if and only if  $\beta$  is unimodular, that is, if and only if  $\phi$  is an isomorphism.