Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ARITHMETIC OF BINARY CUBIC FORMS
Autor: HOFFMAN, J. William / MORALES, Jorge
Kapitel: 4. A Lie algebra representation
DOI: https://doi.org/10.5169/seals-64795

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

4. A Lie algebra representation

Let M be a projective R-module of rank two. Let $G = \mathrm{Aut}_R(M)$ and let $\mathfrak{g} = \mathrm{End}_R(M)$ viewed as a Lie algebra over R.

The group G acts on the right on $\mathrm{Sym}_R(M^*)$ by algebra automorphisms via

$$(F\sigma)(\mathbf{x}) = F(\sigma\mathbf{x})$$

for $F \in \mathrm{Sym}_R(M^*)$ and $\sigma \in G$. Taking the formal derivative at the origin of the associated map

$$G \to \mathrm{Aut}_{R-\mathrm{alg}}(\mathrm{Sym}_R(M^*))$$

we get a representation of Lie algebras

$$\rho\colon \mathfrak{g} \longrightarrow \mathrm{Der}_R(\mathrm{Sym}_R(M^*))\,. \tag{22}$$

The action of G preserves the homogeneous components $\mathrm{Sym}_R^n(M^*)$ and also the submodule $S^n(M^*)$ of Gaussian forms. The same is true for the Lie algebra action of $\mathfrak{g}$.

We shall compute the action of $\mathfrak{g}$ on $S^n(M^*)$ explicitly:

Lemma 4.1. *Let $F \in S^n(M^*)$ and let T be the associated n-linear form. Then*

$$\rho(g)(F)(\mathbf{x}) = nT(g\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$$

for all $g \in \mathfrak{g}$.

Proof. To compute the derivative of $G \to \mathrm{Aut}_R(S^n(M^*))$, we extend the scalars to the "dual numbers" $R[\epsilon]/(\epsilon^2)$. Using the symmetry of T we have

$$F((1 + g\epsilon)\mathbf{x}) = F(\mathbf{x}) + nT(g\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})\,\epsilon\,,$$

which proves our assertion. □

Let C/R be a quadratic algebra in the sense of Section 2 and let M be an invertible C-module. Then we have a natural map $C \to \mathrm{End}_R(M)$ and we can restrict the representation ρ to C. Note that when R is a field and C is an étale quadratic algebra then the image of C is a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}$.

Comparing (22) with equation (21), we see that the C-module structure on $S_C^3(M^*)$ is related to the Lie algebra action by

$$cF = \tfrac{1}{3}\rho(c)F\,. \tag{23}$$

We will make this explicit in a special case that we need:

LEMMA 4.2. *Let $F \in S^3(M^*)$ be a binary cubic form over a field K of characteristic not 2 or 3. Let q_F be its determining form, and $C = C^+(q_F)$ its even Clifford algebra. Let x_1, x_2 be coordinates on the vector space M with respect to a basis $\mathbf{m}_1, \mathbf{m}_2$. Let*

$$\tau = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 \in C = C^+(q_F).$$

Note that $\tau^2 = D$ is the discriminant of q_F. Then

$$\rho(\tau) = \frac{\partial q_F}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial q_F}{\partial x_1}\frac{\partial}{\partial x_2},$$

acting on forms of any degree.

Proof. As we have seen,

$$q_F(x_1\mathbf{m}_1 + x_2\mathbf{m}_2) = Px_1^2 + Qx_1x_2 + Rx_2^2,$$

where $P = a_1^2 - a_0a_2$, $Q = a_1a_2 - a_0a_3$, and $R = a_2^2 - a_1a_3$. By direct computation in the Clifford algebra C, we see that

$$\tau\mathbf{m}_1 = Q\mathbf{m}_1 - 2P\mathbf{m}_2$$
$$\tau\mathbf{m}_2 = 2R\mathbf{m}_1 - Q\mathbf{m}_2.$$

Since $\rho(c)$ is a derivation of $\mathrm{Sym}_R(M^*)$, we have

$$\rho(c) = \rho(c)(x_1)\frac{\partial}{\partial x_1} + \rho(c)(x_2)\frac{\partial}{\partial x_2}.$$

Thus $\tau(x_1\mathbf{m}_1 + x_2\mathbf{m}_2) = (Qx_1 + 2Rx_2)\mathbf{m}_1 - (2Px_1 + Qx_2)\mathbf{m}_2$, which gives $\rho(\tau)(x_1) = \partial q_F/\partial x_2$ and $\rho(\tau)(x_2) = -\partial q_F/\partial x_1$. □

COROLLARY 4.3. $\rho(\tau)q_F = 0$ *and*

$$\rho(\tau)F = \begin{vmatrix} \partial F/\partial x_1 & \partial F/\partial x_2 \\ \partial q_F/\partial x_1 & \partial q_F/\partial x_2 \end{vmatrix} \tag{24}$$
$$= 3G_F,$$

where G_F is as in (5).

REMARK 4.4. If we further assume that C is an étale algebra, then as we have remarked, ρ maps C onto a Cartan subalgebra of $\mathrm{End}_K(M) \sim \mathfrak{gl}(2, K)$. This algebra decomposes as

$$\mathfrak{h}_C = \mathfrak{z} \oplus \mathfrak{h}'_C$$

where the first factor is the center, consisting of scalar matrices, and the second factor is the intersection $\mathfrak{h}_C \cap \mathfrak{sl}(2, K)$, consisting of matrices of trace 0. As the formulas in the proof of the preceding lemma show that τ acts on M with trace 0, we see that $\mathfrak{h}'_C = K\tau$.

THEOREM 4.5. *Let C/R be a quadratic algebra such that $C \otimes K$ is étale over K. Let M be a projective rank-one C-module and let $F \in S^3(M^*)$ be such that the determining mapping q_F is not 0. Then the following conditions are equivalent:*

(a) *F is a C-form*

(b) *$(M, q_F, \mathcal{D}(M))$ is of type C*

(c) *$\rho(c)\rho(\overline{c})F = 9n(c)F$ for all $c \in C$.*

Proof. (a)$\Rightarrow$(b). If T is the trilinear form attached to F, then, using the symmetry of $T(c\mathbf{x}, \mathbf{y}, \mathbf{z})$, we have

$$\begin{aligned} q_F(c\mathbf{x}) &= \wedge^2 T(c\mathbf{x}, -, -) \\ &= \wedge^2 (T(\mathbf{x}, c-, -)) \\ &= n(c) \wedge^2 (T(\mathbf{x}, -, -)) \\ &= n(c) q_F(\mathbf{x}), \end{aligned}$$

which proves the claim. In fact, this implication does not depend on $C \otimes K$ being étale.

It is enough to prove the theorem for the case where $R = K$ is a separably closed field. We can assume in this case $C = K[\sigma]$ with σ satisfying $\sigma^2 = 1$. We will make these assumptions for the rest of the proof.

(b)$\Rightarrow$(c). Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M over K with $\sigma\mathbf{m}_1 = \mathbf{m}_1$ and $\sigma\mathbf{m}_2 = -\mathbf{m}_2$. With respect to this basis, the form q_F, being of type C, must have the shape

$$q_F(\mathbf{x}) = \alpha x_1 x_2,$$

where $\alpha \neq 0$. To see that this is so, note that because q_F is of type C, we have $q_F(\sigma\mathbf{m}_1) = n(\sigma)q_F(\mathbf{m}_1) = -q_F(\mathbf{m}_1)$, which shows that $q_F(\mathbf{m}_1) = 0$. One sees similarly that $q_F(\mathbf{m}_2) = 0$. Then the coefficients of $F(\mathbf{x}) = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$ satisfy the relations: $a_1^2 - a_0a_2 = 0$, $a_1a_2 - a_0a_3 = \alpha$ and $a_2^2 - a_1a_3 = 0$. Since $\alpha \neq 0$, it follows at once that $a_1 = a_2 = 0$, so F is of the form $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$. Since $q_F \neq 0$ (in fact nondegenerate under the étaleness hypothesis), the algebra C can be identified with the even Clifford algebra $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. Under that identification we have $\tau = \alpha\sigma$, where τ is defined as in Lemma 4.2. From that lemma we get $\rho(\sigma) = x_1\partial/\partial x_1 - x_2\partial/\partial x_2$, which can be seen directly, since both sides agree on x_1, x_2. Hence $\rho(\sigma)(x_1^{3-i}x_2^i) = (3-2i)x_1^{3-i}x_2^i$. In particular, for $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$ we have

$$\rho(\sigma)\rho(\overline{\sigma})F = -\rho(\sigma)^2F = -9F = 9n(\sigma)F.$$

The more general identity $\rho(c)\rho(\bar{c})F = 9n(c)F$ for $c \in C$ follows from this particular case by noting that, from Lemma 4.1, $\rho(1)F = 3F$.

(c)$\Rightarrow$(a). Suppose that $\rho(\sigma)^2 F = 9F$. Then F must have the form $F = \lambda x_1^3 + \mu x_2^3$. This is because, as we saw in the discussion above, the monomials $x_1^{3-i}x_2^i$ are eigenvectors for the operator $\rho(\sigma)^2$ with eigenvalue $(3-2i)^2$, which equals 9 only for $i = 0$ and $i = 3$. Hence the associated trilinear form is $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 + \mu x_2 y_2 z_2$. Thus $T(\sigma\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 - \mu x_2 y_2 z_2$, which is visibly symmetric in $\mathbf{x}, \mathbf{y}, \mathbf{z}$. $\square$

REMARK 4.6. It is interesting to notice that the syzygy (6) can be recovered from Part (c) of Theorem 4.5. Assume for simplicity that $R = K$ is a field and C is an étale K-algebra. Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M. Let $\tau = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 \in C = C^+(q_F)$ as in Lemma 4.2. As we noted in Remark 4.4, τ generates the trace 0 part of the Cartan subalgebra defined by C. Using the derivation property and Corollary 4.3, we see $\rho(\tau)(G_F^2 - DF^2) = (2/3)(\rho(\tau)^2 F - 9DF)G_F$. From the above theorem, $\rho(\tau)^2 F = 9DF$, so this is 0. On the other hand, $\rho(\tau)q_F = 0$, also by Corollary 4.3, which implies that $\rho(\tau)q_F^3 = 0$. Hence both q_F^3 and $G_F^2 - DF^2$ lie in the subspace on weight 0 (for the action of the Cartan subalgebra $\mathfrak{h}'_C \subset \mathfrak{sl}(2, K)$) of $S^6(M^*)$. As $S^6(M^*)$ is an irreducible representation of $\mathfrak{sl}(2, K)$, this is one-dimensional. Hence q_F^3 and $G_F^2 - D_F F^2$ differ by a constant multiple. A priori, this constant could depend on F (e.g., D). That this is not so can be seen by noting that both sides are of the same degree in the coefficients of F.

COROLLARY 4.7. *Let M be a projective R-module of rank 2, and let $F \in S^3(M^*)$.*

(i) *Let $C = C^+(M, q_F, \mathcal{D}(M))$ and suppose that $C \otimes K$ is étale, and that q_F is primitive. Then F is a C-form.*

(ii) *If F is a C-form for a quadratic R-algebra C and $(M, q_F, \mathcal{D}(M))$ is primitive, then $C \cong C^+(M, q_F, \mathcal{D}(M))$.*

Proof. (i) By Proposition 2.8, $(M, q_F, \mathcal{D}(M))$ is of type C. We conclude by Theorem 4.5.

(ii) If F is a C-form, then by Theorem 4.5, the quadratic mapping $(M, q_F, \mathcal{D}(M))$ is type C. But assuming furthermore that $(M, q_F, \mathcal{D}(M))$ is primitive, we see that C is isomorphic with $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. $\square$

LEMMA 4.8. *Suppose that $C \otimes K$ is étale over K and let (M, F) and (M', F') be cubic C-forms. Assume that the determining mappings $q_F, q_{F'}$ are nonzero. Then every R-linear isomorphism $f\colon (M, F) \to (M', F')$ is either C-linear or C-sesquilinear.*

Proof. The map f will induce an isomorphism of determining quadratic mappings of type C. We conclude by Proposition 2.3. □

5. STRUCTURE OF THE CUBIC C-FORMS

We shall describe below the C-module structure of $S^3_C(M^*)$ and the corresponding C-isomorphism classes.

THEOREM 5.1. *Let M be a rank-one projective C-module. For each $\phi \in \mathrm{Hom}_C(M_C^{\otimes 3}, C^*)$ we define a cubic form by $F_\phi(\mathbf{x}) = \phi(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(1)$. Then*

(i) *The correspondence $\phi \mapsto F_\phi$ is an isomorphism of C-modules $\mathrm{Hom}_C(M_C^{\otimes 3}, C^*) \to S^3_C(M^*)$.*

(ii) *The determining mapping q_{F_ϕ} is primitive if and only if ϕ is an isomorphism.*

(iii) *Two cubic C-forms F and F_1 on M are equivalent over C if and only if there exists $c \in C^\times$ such that $F_1 = c^3 F$.*

Proof. (i) This is a restatement of Proposition 3.7. The map $\phi \mapsto F_\phi$ is a C-isomorphism by definition of the structure of C-module on $S^3_C(M^*)$ in Section 3.

(ii) It is enough to prove our assertion locally, so we assume that M is free over C. Write $M = C\mathbf{m}$ for some $\mathbf{m} \in M$. Let $\lambda = \phi(\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m})$. Then we have $\phi(x\mathbf{m} \otimes y\mathbf{m} \otimes z\mathbf{m}) = \lambda(xyz)$. Let $\beta(y\mathbf{m}, z\mathbf{m}) = \lambda(yz)$ and observe that λ is a basis of C^* over C if and only if the symmetric bilinear form β is unimodular. We have

$$\begin{aligned} q_{F_\phi}(x\mathbf{m}) &= n(x) q_{F_\phi}(\mathbf{m}) \\ &= n(x) \wedge^2 \beta . \end{aligned}$$

It follows from this equality that q_{F_ϕ} is primitive if and only if β is unimodular, that is, if and only if ϕ is an isomorphism.