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Autor: BUCH, Anders Skovsted / Fulton, William
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APPENDIX. A BIJECTION BETWEEN HIVES AND
LITTLEWOOD-RICHARDSON SKEW TABLEAUX
(by William FULTON)

The aim of this appendix is to give a simple and direct bijection between the hives with given boundary (given by a triple of partitions), and the set of Littlewood-Richardson skew tableaux for the given triple. In principle one could construct such a mapping from [4], but it is simpler to do it directly from hives; in the description we give here, it is easy to see that the map is a bijection, without knowing that the two sets have the same cardinality. As in [4], we produce contratableaux, but there is a standard bijection between these and the original Littlewood-Richardson skew tableaux.

Consider an integral hive, with sides having $n + 1$ entries, corresponding to partitions λ , μ , and ν , with $|\nu| = |\lambda| + |\mu|$. The differences down the northwest to southeast border give the partition λ , the differences across the bottom border from right to left give μ , and the differences down the northeast to southwest border give ν (see Theorem 1). The main idea for constructing a skew tableau with a reverse-lattice word is to use the other northwest to southeast rows of entries to construct a chain of subpartitions of λ .

The entries of the hive will be denoted a_k^i , with $1 \leq i \leq n + 1$ and $0 \leq k \leq n + 1 - i$. Here the superscript denotes the northwest to southeast row of the entry, with the first row being the long row on the boundary, and the others in order below that; the subscripts number the entries along the rows, from northwest to southeast.

$$\begin{array}{cccc}
 & & a_0^1 & \\
 & & & \\
 & a_0^2 & & a_1^1 \\
 & & & \\
 a_0^3 & & a_1^2 & a_2^1 \\
 & & & \\
 a_0^4 & a_1^3 & a_2^2 & a_3^1
 \end{array}$$

Note that $a_0^1 = 0$, and that $\lambda_k = a_k^1 - a_{k-1}^1$ for $1 \leq k \leq n$.

For $1 \leq i \leq n$ define a sequence $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{n+1-i}^{(i)})$ by setting $\lambda_k^{(i)} = a_k^i - a_{k-1}^i$. Note that $\lambda^{(1)} = \lambda$.

There are three types of rhombus inequalities, depending on the orientation of the rhombus. We first consider two of them:



This says that $\lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$.



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Together, (1) and (2) say that $\lambda_k^{(i)} \geq \lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$. In particular, each sequence $\lambda^{(i)}$ is weakly decreasing, and we have a nested sequence of partitions: $\lambda^{(1)} \supset \lambda^{(2)} \supset \dots \supset \lambda^{(n)} \supset \lambda^{(n+1)} = \emptyset$.

For example, the hive

		0		
		10	6	
	17	14	10	
24	21	18	14	
28	26	23	19	15

gives the chain of partitions $(6, 4, 4, 1) \supset (4, 4, 1) \supset (4, 2) \supset (2) \supset \emptyset$.

We identify partitions with Young diagrams, but rotated by 180 degrees, so the diagram for a partition λ has λ_k boxes in the k^{th} row from the bottom, and the rows are lined up on the right. Fill the boxes by putting the integer i in each box of $\lambda^{(i)} - \lambda^{(i+1)}$. The conditions (1) and (2) say exactly that the result T is a skew tableau on this shape, that is, it is weakly increasing

across rows and strictly increasing down columns. Such a T is often called a contratableau of shape λ . In our example, T is

					1
		1	1	1	2
		2	2	3	3
1	1	3	3	4	4

The word $w(T)$ is obtained by reading from left to right in rows, from bottom to top. In the example, $w(T) = 113344223311121$.

Let $U(\mu)$ be the tableau of shape μ whose i^{th} row has μ_i entries, all equal to i . The word $w(U(\mu))$ is similarly read from left to right, bottom to top. In our example, $\mu = (4, 4, 3, 2)$, and $w(U(\mu)) = 443332221111$.

Now we consider the last rhombus inequalities:

$$(3) \quad \begin{array}{ccccc} & & a_{k-1}^i & & a_k^{i-1} \\ & & \bullet & \text{---} & \bullet \\ & \swarrow & & \searrow & \\ a_{k-1}^{i+1} & & & & a_k^i \\ \bullet & & & & \bullet \end{array}$$

These say that $a_{k-1}^{i+1} - a_{k-1}^i \leq a_k^i - a_k^{i-1}$. We claim that this is equivalent to the condition that $w(T) \cdot w(U(\mu))$ is a reverse lattice word [5, §5.2].

This asserts that, if we divide this word at any point, the number of times that i occurs to the right of this point does not exceed the number of times that $i-1$ occurs to the right of this point. We only need to check this at a division corresponding to the place in the k^{th} row from the bottom of T that divides elements strictly smaller than i from elements greater than or equal to i . The number of times that i occurs here is

$$\begin{aligned} & (\lambda_k^{(i)} - \lambda_k^{(i+1)}) + (\lambda_{k+1}^{(i)} - \lambda_{k+1}^{(i+1)}) + \cdots + (\lambda_{n+1-i}^{(i)} - 0) + \mu_i \\ &= (\lambda_k^{(i)} + \lambda_{k+1}^{(i)} + \cdots + \lambda_{n+1-i}^{(i)}) - (\lambda_k^{(i+1)} + \lambda_{k+1}^{(i+1)} + \cdots + \lambda_{n-i}^{(i+1)}) + \mu_i \\ &= (a_{n+1-i}^i - a_{k-1}^i) - (a_{n-i}^{i+1} - a_{k-1}^{i+1}) + (a_{n-i}^{i+1} - a_{n+1-i}^i) \\ &= a_{k-1}^{i+1} - a_{k-1}^i. \end{aligned}$$

Similarly, the number of times that $i-1$ occurs is

$$(\lambda_{k+1}^{(i-1)} - \lambda_{k+1}^{(i)}) + (\lambda_{k+2}^{(i-1)} - \lambda_{k+2}^{(i)}) + \cdots + (\lambda_{n+2-i}^{(i-1)} - 0) + \mu_{i-1} = a_k^i - a_k^{i-1}.$$

Note that the number of times i occurs in all of T is $a_0^{i+1} - a_0^i - \mu_i = \nu_i - \mu_i$.

This process is reversible. Given any contratableau T of shape λ such that $w(T) \cdot w(U(\mu))$ is a reverse lattice word, T determines the chain

$\lambda^{(1)} \supset \lambda^{(2)} \supset \dots \supset \lambda^{(n)} \supset \emptyset$, and from these partitions one successively fills in the entries in the northwest to southeast diagonal rows of the hive; the rhombus inequalities (1)–(3) are automatically satisfied.

To make the story complete, we recall why such contratableaux correspond to Littlewood-Richardson skew tableaux, using standard results about tableaux, as in [5]. However, it may be pointed out that these contratableaux are at least as easy to produce and enumerate as the more classical skew tableaux. First, the condition that $w(T) \cdot w(U(\mu))$ is a reverse lattice word, given that the number of times i occurs in T is $\nu_i - \mu_i$, is equivalent to asserting that $w(T) \cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$ [5, §5.2]. The rectification R of a contratableau T of shape λ is easily seen to be a tableau of shape λ , and with the same property that $w(R) \cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$. The correspondence between tableaux and contratableaux of shape λ is a bijection, by reversing the rectification process.

Now the condition that $w(R) \cdot w(U(\mu))$ be Knuth equivalent to $w(U(\nu))$ is equivalent to the condition that $R \cdot U(\mu) = U(\nu)$ in the plactic monoid of tableaux [5, §2.1]. It is easy to see, from the definition of multiplying tableaux by column bumping entries of the first tableau into the second [5, §A.2], that if R and S are tableaux with $R \cdot S = U(\beta)$, then S must be equal to $U(\alpha)$ for some partition α . This gives a correspondence between the set of tableaux R that we are looking at and the set of pairs (R, S) with R of shape λ , S of shape μ , whose product is the tableau $U(\nu)$. There is a standard construction [5, §5.1] between these pairs and the set of skew tableau on the shape ν/λ of content μ whose word is a reverse-lattice word.

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