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APPENDIX. A BIJECTION BETWEEN HIVES AND  
LITTLEWOOD-RICHARDSON SKEW TABLEAUX  
(by William FULTON)

The aim of this appendix is to give a simple and direct bijection between the hives with given boundary (given by a triple of partitions), and the set of Littlewood-Richardson skew tableaux for the given triple. In principle one could construct such a mapping from [4], but it is simpler to do it directly from hives; in the description we give here, it is easy to see that the map is a bijection, without knowing that the two sets have the same cardinality. As in [4], we produce contratableaux, but there is a standard bijection between these and the original Littlewood-Richardson skew tableaux.

Consider an integral hive, with sides having  $n + 1$  entries, corresponding to partitions  $\lambda$ ,  $\mu$ , and  $\nu$ , with  $|\nu| = |\lambda| + |\mu|$ . The differences down the northwest to southeast border give the partition  $\lambda$ , the differences across the bottom border from right to left give  $\mu$ , and the differences down the northeast to southwest border give  $\nu$  (see Theorem 1). The main idea for constructing a skew tableau with a reverse-lattice word is to use the other northwest to southeast rows of entries to construct a chain of subpartitions of  $\lambda$ .

The entries of the hive will be denoted  $a_k^i$ , with  $1 \leq i \leq n + 1$  and  $0 \leq k \leq n + 1 - i$ . Here the superscript denotes the northwest to southeast row of the entry, with the first row being the long row on the boundary, and the others in order below that; the subscripts number the entries along the rows, from northwest to southeast.

$$\begin{array}{cccc}
 & & & a_0^1 \\
 & & & \\
 & & a_0^2 & a_1^1 \\
 & & & \\
 & a_0^3 & a_1^2 & a_2^1 \\
 & & & \\
 a_0^4 & a_1^3 & a_2^2 & a_3^1
 \end{array}$$

Note that  $a_0^1 = 0$ , and that  $\lambda_k = a_k^1 - a_{k-1}^1$  for  $1 \leq k \leq n$ .

For  $1 \leq i \leq n$  define a sequence  $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{n+1-i}^{(i)})$  by setting  $\lambda_k^{(i)} = a_k^i - a_{k-1}^i$ . Note that  $\lambda^{(1)} = \lambda$ .

There are three types of rhombus inequalities, depending on the orientation of the rhombus. We first consider two of them:



This says that  $\lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$ .



This says that  $\lambda_k^{(i)} \geq \lambda_k^{(i+1)}$ .

Together, (1) and (2) say that  $\lambda_k^{(i)} \geq \lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$ . In particular, each sequence  $\lambda^{(i)}$  is weakly decreasing, and we have a nested sequence of partitions:  $\lambda^{(1)} \supset \lambda^{(2)} \supset \dots \supset \lambda^{(n)} \supset \lambda^{(n+1)} = \emptyset$ .

For example, the hive

		0		
		10	6	
	17	14	10	
24	21	18	14	
28	26	23	19	15

gives the chain of partitions  $(6, 4, 4, 1) \supset (4, 4, 1) \supset (4, 2) \supset (2) \supset \emptyset$ .

We identify partitions with Young diagrams, but rotated by 180 degrees, so the diagram for a partition  $\lambda$  has  $\lambda_k$  boxes in the  $k^{\text{th}}$  row from the bottom, and the rows are lined up on the right. Fill the boxes by putting the integer  $i$  in each box of  $\lambda^{(i)} - \lambda^{(i+1)}$ . The conditions (1) and (2) say exactly that the result  $T$  is a skew tableau on this shape, that is, it is weakly increasing

across rows and strictly increasing down columns. Such a  $T$  is often called a contratableau of shape  $\lambda$ . In our example,  $T$  is

					1
		1	1	1	2
		2	2	3	3
1	1	3	3	4	4

The word  $w(T)$  is obtained by reading from left to right in rows, from bottom to top. In the example,  $w(T) = 113344223311121$ .

Let  $U(\mu)$  be the tableau of shape  $\mu$  whose  $i^{\text{th}}$  row has  $\mu_i$  entries, all equal to  $i$ . The word  $w(U(\mu))$  is similarly read from left to right, bottom to top. In our example,  $\mu = (4, 4, 3, 2)$ , and  $w(U(\mu)) = 443332221111$ .

Now we consider the last rhombus inequalities:

$$(3) \quad \begin{array}{ccccc} & & a_{k-1}^i & & a_k^{i-1} \\ & & \bullet & \text{---} & \bullet \\ & \swarrow & & \searrow & \\ a_{k-1}^{i+1} & & & & a_k^i \\ \bullet & & & & \bullet \end{array}$$

These say that  $a_{k-1}^{i+1} - a_{k-1}^i \leq a_k^i - a_k^{i-1}$ . We claim that this is equivalent to the condition that  $w(T) \cdot w(U(\mu))$  is a reverse lattice word [5, §5.2].

This asserts that, if we divide this word at any point, the number of times that  $i$  occurs to the right of this point does not exceed the number of times that  $i-1$  occurs to the right of this point. We only need to check this at a division corresponding to the place in the  $k^{\text{th}}$  row from the bottom of  $T$  that divides elements strictly smaller than  $i$  from elements greater than or equal to  $i$ . The number of times that  $i$  occurs here is

$$\begin{aligned} & (\lambda_k^{(i)} - \lambda_k^{(i+1)}) + (\lambda_{k+1}^{(i)} - \lambda_{k+1}^{(i+1)}) + \cdots + (\lambda_{n+1-i}^{(i)} - 0) + \mu_i \\ &= (\lambda_k^{(i)} + \lambda_{k+1}^{(i)} + \cdots + \lambda_{n+1-i}^{(i)}) - (\lambda_k^{(i+1)} + \lambda_{k+1}^{(i+1)} + \cdots + \lambda_{n-i}^{(i+1)}) + \mu_i \\ &= (a_{n+1-i}^i - a_{k-1}^i) - (a_{n-i}^{i+1} - a_{k-1}^{i+1}) + (a_{n-i}^{i+1} - a_{n+1-i}^i) \\ &= a_{k-1}^{i+1} - a_{k-1}^i. \end{aligned}$$

Similarly, the number of times that  $i-1$  occurs is

$$(\lambda_{k+1}^{(i-1)} - \lambda_{k+1}^{(i)}) + (\lambda_{k+2}^{(i-1)} - \lambda_{k+2}^{(i)}) + \cdots + (\lambda_{n+2-i}^{(i-1)} - 0) + \mu_{i-1} = a_k^i - a_k^{i-1}.$$

Note that the number of times  $i$  occurs in all of  $T$  is  $a_0^{i+1} - a_0^i - \mu_i = \nu_i - \mu_i$ .

This process is reversible. Given any contratableau  $T$  of shape  $\lambda$  such that  $w(T) \cdot w(U(\mu))$  is a reverse lattice word,  $T$  determines the chain

$\lambda^{(1)} \supset \lambda^{(2)} \supset \dots \supset \lambda^{(n)} \supset \emptyset$ , and from these partitions one successively fills in the entries in the northwest to southeast diagonal rows of the hive; the rhombus inequalities (1)–(3) are automatically satisfied.

To make the story complete, we recall why such contratableaux correspond to Littlewood-Richardson skew tableaux, using standard results about tableaux, as in [5]. However, it may be pointed out that these contratableaux are at least as easy to produce and enumerate as the more classical skew tableaux. First, the condition that  $w(T) \cdot w(U(\mu))$  is a reverse lattice word, given that the number of times  $i$  occurs in  $T$  is  $\nu_i - \mu_i$ , is equivalent to asserting that  $w(T) \cdot w(U(\mu))$  is Knuth equivalent to  $w(U(\nu))$  [5, §5.2]. The rectification  $R$  of a contratableau  $T$  of shape  $\lambda$  is easily seen to be a tableau of shape  $\lambda$ , and with the same property that  $w(R) \cdot w(U(\mu))$  is Knuth equivalent to  $w(U(\nu))$ . The correspondence between tableaux and contratableaux of shape  $\lambda$  is a bijection, by reversing the rectification process.

Now the condition that  $w(R) \cdot w(U(\mu))$  be Knuth equivalent to  $w(U(\nu))$  is equivalent to the condition that  $R \cdot U(\mu) = U(\nu)$  in the plactic monoid of tableaux [5, §2.1]. It is easy to see, from the definition of multiplying tableaux by column bumping entries of the first tableau into the second [5, §A.2], that if  $R$  and  $S$  are tableaux with  $R \cdot S = U(\beta)$ , then  $S$  must be equal to  $U(\alpha)$  for some partition  $\alpha$ . This gives a correspondence between the set of tableaux  $R$  that we are looking at and the set of pairs  $(R, S)$  with  $R$  of shape  $\lambda$ ,  $S$  of shape  $\mu$ , whose product is the tableau  $U(\nu)$ . There is a standard construction [5, §5.1] between these pairs and the set of skew tableau on the shape  $\nu/\lambda$  of content  $\mu$  whose word is a reverse-lattice word.

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