

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FINITE GROUP ACTIONS ON THE 7-SPHERE  
**Autor:** Geiges, Hansjörg / THOMAS, Charles B.  
**DOI:** <https://doi.org/10.5169/seals-64810>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 05.12.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## FINITE GROUP ACTIONS ON THE 7-SPHERE

by Hansjörg GEIGES and Charles B. THOMAS

The aim of this note is to prove the following theorem.

THEOREM. (i) *Let  $\pi$  be a finite group acting freely and topologically on  $S^7$ . Then  $\pi$  can also act freely and linearly on  $S^7$ .*

(ii) *For odd natural numbers  $2s - 1$  different from 1, 3 or 7, one can always find a free smooth action on  $S^{2s-1}$  by a finite group which cannot act freely and linearly.*

A weaker version of this theorem was announced in [3]. The proof of the theorem uses only classical results about finite group actions on spheres, mostly from [15], and it is a little surprising that this theorem has not been observed before. The interest of this result in the context of more recent investigations on geometric structures on spherical space forms is explained in [3] and [4].

The (as yet unproven) analogue of part (i) of our theorem in dimension 3 forms an essential step in Thurston's geometrisation programme [13]. According to Thurston's conjecture, any 3-manifold covered by  $S^3$  is actually a quotient of  $S^3$  under a free linear action of some finite group  $\pi$ . This splits into proving the said analogue of our theorem, and then showing that the only possible actions of  $\pi$  are indeed linear, i.e. given by fixed-point free representations of  $\pi$  in  $\mathrm{SO}(4)$ . A proof of the former was announced in [10], but at present it is not clear whether all details of the argument can be filled in. For some groups acting on  $S^3$  it is known that they can only act linearly, cf. [4]. See [5, Problems 3.37 and 3.45] for further references on this issue.

The only finite groups acting freely and topologically on  $S^1$  are the cyclic groups, and any such action is conjugate to a linear one.

Notice that part (i) of our theorem does not make any statement about the possible actions of the groups  $\pi$  on  $S^7$ . As is well-known, and contrary to what one hopes to be true in dimension 3, there are many exotic smooth actions (i.e. actions not conjugate to a linear one) on spheres of dimension  $\geq 5$ , even by cyclic groups, see [14, Chapter 14].

The proof of part (i) of the theorem is case by case and exploits the discussion of groups with periodic cohomology (or periodic groups, for short) in [15]. First we collect some fundamental and well-known facts. Recall that a finite group  $\pi$  is said to satisfy the  $pq$ -condition ( $p, q$  prime) if all subgroups of  $\pi$  of order  $pq$  are cyclic.

Since the group  $\pi$  acts freely on  $S^7$ , it has periodic cohomology, and the (minimal) cohomological period divides 8. As a periodic group,  $\pi$  satisfies all  $p^2$ -conditions, cf. [1, Thm. VI.9.5]. By [8] the fact that  $\pi$  acts freely on some sphere implies that all  $2p$ -conditions are satisfied; there are no free actions by a dihedral group on  $S^{2s-1}$ .

A group of order  $pq$  with  $p$  and  $q$  distinct odd primes must be either the cyclic group  $C_{pq}$  or the metacyclic group  $D_{p,q}$  given as a nontrivial extension of  $C_p$  by  $C_q$ . But  $D_{p,q}$  has cohomological period equal to  $2q$ , cf. [2, p. 229], so a group  $\pi$  as in part (i) of our theorem also satisfies all  $pq$ -conditions. Recall from [15, Thm. 5.3.1] that the  $pq$ -conditions are necessary for the existence of a free linear action on some sphere.

We now distinguish between the solvable and non-solvable cases.

**A.** If  $\pi$  is solvable, then Theorem 6.1.11 of Wolf's monograph [15] applies. This theorem gives a complete list of the finite solvable groups with periodic cohomology, separated into four classes, and states that such a group satisfies all  $pq$ -conditions if and only if  $\pi$  can act freely and linearly on some sphere. As it stands, however, it is not strong enough to guarantee the existence of a free linear action on  $S^7$ , so we need to take a closer look at the four classes in Wolf's classification theorem.

**I.** The first class consists of the metacyclic groups of order  $mn$  with presentation

$$\{A, B \mid A^m = B^n = 1, \quad BAB^{-1} = A^r\},$$

where  $m \geq 1$ ,  $n \geq 1$ ,  $\gcd(n(r-1), m) = 1$ , and  $r^n \equiv 1 \pmod{m}$ . Furthermore, the fact that  $\pi$  satisfies all  $pq$ -conditions is equivalent to the following: if  $d$

is the smallest natural number such that  $r^d \equiv 1 \pmod{m}$ , then  $n/d$  is divisible by every prime divisor of  $d$ . This numerical condition will be satisfied in all solvable cases **II** to **IV** below.

The cohomological period of  $\pi$  is  $2d$  (cf. [2, p.229]), so we must have  $d \in \{1, 2, 4\}$ . Furthermore, Wolf [15, Thm. 5.5.1] gives explicit fixed-point free linear representations of (real) degree  $2d$ . Recall that a linear representation  $\rho: \pi \rightarrow \text{GL}(2d)$  is called *fixed-point free* if  $\rho(g)$  does not have 1 for an eigenvalue for  $1 \neq g \in \pi$ . Furthermore, a real linear representation is always equivalent to an orthogonal representation, so fixed-point free representations of degree  $2d$  induce a free linear action on  $S^{2d-1}$  (and of course also on  $S^{2dk-1}$  for any positive integer  $k$ ). In fact, one can always obtain unitary representations, cf. [16].

Notice that if  $\pi$  is a metacyclic group of odd order, then the only possibility for  $d$  is that it equal 1, which means that  $\pi$  is actually a cyclic group.

**II.** The second class of solvable groups  $\pi$  admit a presentation with generators  $A, B, R$ , relations as in **I**, and additional relations

$$R^2 = B^{n/2}, \quad RAR^{-1} = A^l, \quad RBR^{-1} = B^k,$$

subject to the numerical conditions as in **I** and the further conditions

$$\begin{aligned} l^2 &\equiv r^{k-1} \equiv 1 \pmod{m}, & n &= 2^u v, \quad u \geq 2, \quad v \text{ odd}, \\ k &\equiv -1 \pmod{2^u}, & k^2 &\equiv 1 \pmod{n}. \end{aligned}$$

The order of this group is  $2mn$ . With  $d$  defined as in **I**, again we must have  $d \in \{1, 2, 4\}$ , since the subgroup generated by  $A$  and  $B$  is of type **I** with cohomological period  $2d$ . According to [15, p.180], the group  $\pi$  admits a fixed-point free representation of real dimension  $4d$ . So we only need to show that the case  $d = 4$  cannot arise.

Indeed, on the one hand the numerical conditions stipulate  $2^u \mid k+1$ , so  $4 \mid k+1$ . On the other hand, since  $r^{k-1} \equiv 1 \pmod{m}$ , we also know that  $d \mid k-1$ . Clearly this is impossible for  $d = 4$ .

We now want to identify such a group  $\pi$  of type **II** with one of the groups  $C_{m_1} \times Q(2^{u+1}m_2, m_3, m_4)$  described by Milnor [8], with the  $m_i$  odd and pairwise coprime. We only summarise the observations necessary to make this identification.



The subgroup of  $\pi$  generated by  $R$  and  $B^v$  is isomorphic to the quaternion group  $Q(2^{u+1})$ . The normal subgroup of  $\pi$  generated by  $A$  and  $B^{2^u}$  is cyclic of odd order  $mv$ , and we have a split extension

$$C_{mv} \twoheadrightarrow \pi \twoheadrightarrow Q(2^{u+1}).$$

The generators  $R$  and  $B^v$  of  $Q(2^{u+1})$  act on  $C_{mv}$  as a pair of commuting involutions, so we get a splitting

$$C_{mv} = C_{m_1} \times C_{m_2} \times C_{m_3} \times C_{m_4}$$

with the  $m_i$  pairwise coprime and such that

- $R$  and  $B^v$  act trivially on  $C_{m_1}$ ,
- $B^v$  inverts elements in  $C_{m_2}$  and  $C_{m_3}$ ,
- $R$  inverts elements in  $C_{m_2}$  and  $C_{m_4}$ .

This identifies  $\pi$  as a Milnor group as described above, cf. [2, p.229].

Fixed-point free representations of the groups  $Q(2^{u+1}m_2, m_3, m_4)$  in  $SU(4)$  are described in [2, p.255]. Taking the product with a group of coprime order never constitutes a problem: if  $\rho$  is a fixed-point free unitary representation of some group  $\pi$ , then

$$\tilde{\rho}(t, g) = \exp(2\pi i/m)\rho(g)$$

defines a fixed-point free representation of  $C_m^t \times \pi$  if  $m$  is coprime to  $|\pi|$ .

**III.** The third class of solvable groups described by Wolf has a presentation with generators  $A, B, P, Q$ , relations and numerical conditions as in **I**, and further relations

$$P^4 = 1, \quad P^2 = Q^2 = (PQ)^2, \quad AP = PA, \quad AQ = QA, \\ BPB^{-1} = Q, \quad BQB^{-1} = PQ.$$

Furthermore,  $n$  has to be odd and divisible by 3.

As in the previous cases we have  $d \in \{1, 2, 4\}$ , and from  $d \mid n$  we conclude  $d = 1$ . This implies the relation  $BAB^{-1} = A$ . Write  $n$  in the form  $n = 3^u v$  with  $u \geq 1$  and  $v$  not divisible by 3. Then the subgroup of  $\pi$  generated by  $P, Q$  and  $B^v$  is the generalised binary tetrahedral group  $T_u^*$  of order  $8 \cdot 3^u$ . With  $C_{mv}$  denoting the cyclic group of order  $mv$  generated by  $AB^{3^u}$  we have the split extension

$$C_{mv} \twoheadrightarrow \pi \twoheadrightarrow T_u^*,$$

and one easily verifies that  $AB^{3^u}$  commutes with the generators of  $T_u^*$ . So  $\pi$  is actually a direct product

$$C_{mv} \times T_u^*.$$

Notice that  $m$  must be odd (and also coprime to 3 by the conditions from **I**), otherwise  $\pi$  would contain a subgroup  $C_2 \times C_2$  and would not be periodic. The group  $T_u^*$  is well known to act freely and linearly on  $S^3$ , and hence also on  $S^7$ , and the same is true for  $C_{mv} \times T_u^*$ .

**IV.** For the fourth class of solvable groups we only need to observe from Wolf's explicit presentation that they contain a normal subgroup of type **III** of index 2. Thus one may take a 4-dimensional fixed-point free representation of this normal subgroup, and then the induced 8-dimensional representation of the full group will also be fixed-point free (see [15, Lemma 5.5.3]), since the order of the normal subgroup is divisible by every prime divisor of the order of the full group. The groups in this class can be identified as extensions of a cyclic group (of order coprime to 6) by a generalised binary octahedral group  $O_u^*$ . As observed by Milnor [8], the latter have cohomological period 4 and fixed-point free representation in dimension 8, but none in dimension 4 unless  $u = 1$ .

This concludes the discussion of the solvable cases.

**B.** We now turn to the case that  $\pi$  is non-solvable. Notice that in this case the order of  $\pi$  must be even; otherwise,  $\pi$  being a periodic group, all Sylow subgroups are cyclic, and this would imply that  $\pi$  is solvable of type **I** by an old result of Burnside, cf. [15, Thm. 5.4.1].

By Suzuki's classification of periodic non-solvable groups [11, Thm. C], our  $\pi$  contains a subgroup isomorphic to  $SL_2(r)$  with  $r \geq 5$  prime, the multiplicative group of  $(2 \times 2)$ -matrices of determinant 1 with coefficients in the field of  $r$  elements (recall that  $SL_2(3) = T_1^*$  is solvable). The cohomological period of  $SL_2(r)$  equals  $\text{lcm}(4, r-1)$ , cf. [6, Lemma 1.3]. Since for a group  $\pi$  in our theorem this has to divide 8, the only possibility is  $r = 5$ . According to Suzuki, there are two types to consider.

**V.** The group  $\pi$  is the direct product of  $SL_2(5)$  (which has order 120) and a metacyclic group of order coprime to 30. By our comment at the end of **I**, this metacyclic group has to be cyclic, that is,  $\pi = C_m \times SL_2(5)$  with  $\text{gcd}(m, 30) = 1$ . The group  $SL_2(5)$  is isomorphic to the binary icosahedral group  $I^*$  and has fixed-point free representations of real degree 4, and so the same holds for  $\pi$ .

**VI.** The final possibility is that  $\pi$  is an extension of a group of type **V** by a cyclic group of order 2. So we can argue as in **IV** to get a fixed-point free representation of real degree 8.

This concludes the proof of part (i) of the theorem.

Much of this discussion extends to free actions by finite groups  $\pi$  on  $S^{2^t-1}$ , in which case  $\pi$  must have period dividing  $2^t$ . The result for solvable groups holds unchanged. To be more precise, in case **I** we obtain metacyclic groups of period  $2d$  admitting fixed-point free representations of degree  $2d$ , with  $d \in \{1, 2, 4, \dots, 2^{k-1}\}$ . In the cases **II**, **III** and **IV**, we get exactly the same groups as before (i.e. no higher values of  $d$  occur).

For non-solvable groups the restriction that the cohomological period divide  $2^t$  implies that we only get groups  $SL_2(r)$  with  $r$  a prime of the form  $r = 2^w + 1$  with  $2 \leq w \leq t$ , as well as extensions of these groups as described in **V** and **VI**. According to [12] these groups admit a free topological action on  $S^{2^w-1}$  (and hence on  $S^{2^t-1}$  by the join construction) which is conjugate to a free linear action when restricted to any proper subgroup; cf. the discussion in [4].

We now turn to the proof of part (ii). First suppose that  $2s$  is divisible by some odd prime  $q$  and write  $2s = 2kq$ . Choose an odd prime  $p$  that divides  $2^q - 1$ . By the Fermat-Euler theorem we know that  $2^{p-1} - 1$  is also divisible by  $p$ , so  $q$  has to be a divisor of  $p - 1$ , and in particular  $p$  and  $q$  will be coprime.

We can therefore define a metacyclic (but not cyclic) group  $D_{p,q}$  as in **I** with  $m = p$ ,  $n = q$ , and  $r = 2$ . By [9] this group acts freely and smoothly on  $S^{2q-1}$ , and hence also on  $S^{2s-1}$ , the join of  $k$  copies of  $S^{2q-1}$ .

Finally, if  $2s = 2^k$ ,  $k \geq 4$ , we appeal to the result of [7] that  $SL_2(r)$  acts freely and smoothly on  $S^{2s-1}$  with  $2s = \text{lcm}(4, r-1)$ , that is, on a sphere of the dimension predicted by the cohomological period. In the particular case at hand, we can choose  $r = 17$ . The group  $SL_2(17)$  admits a free smooth action on  $S^{16-1}$  and hence on the join  $S^{2s-1}$  of  $2^{k-4}$  copies of  $S^{15}$ . But the only groups  $SL_2(r)$  that can act freely and linearly on some sphere are  $SL_2(3)$  and  $SL_2(5)$ , see [15, Thm. 6.3.1] – even though the non-solvable groups  $SL_2(r)$ ,  $r \geq 5$  prime, satisfy all  $pq$ -conditions for primes  $r$  of the form  $r = 2^l + 1$  (and only for these), cf. [15, p. 197].

This concludes the proof of part (ii).

The last comment in the proof above and our earlier remarks about actions on  $S^{2^t-1}$  also imply the following: If  $\pi$  acts freely and topologically on  $S^{2^t-1}$ , then  $\pi$  satisfies all  $pq$ -conditions. On  $S^{2^s-1}$  with  $2s \neq 2^t$  there are always free smooth actions by metacyclic groups  $D_{p,q}$  violating the  $pq$ -conditions.

ACKNOWLEDGEMENT. Part (ii) of the theorem was prompted by a question of Raymond Lickorish.

## REFERENCES

- [1] BROWN, K. S. *Cohomology of Groups*. Springer, 1982.
- [2] DAVIS, J. F. and R. J. MILGRAM. A survey of the spherical space form problem. *Mathematical Reports 2*, Part 2. Harwood Academic Publishers, 1985.
- [3] GEIGES, H. and C. B. THOMAS. Contact structures on 7-manifolds. In: *Geometry, Topology, and Dynamics* (F. Lalonde, ed.). CRM Proc. Lecture Notes 15 (1998), 53–67.
- [4] GEIGES, H. and C. B. THOMAS. Almost linear actions by finite groups on  $S^{2n-1}$ . In: *Proceedings of the Kirbyfest* (J. Hass and M. Scharlemann, eds.). Geometry & Topology Monographs 2 (1999), 135–156.
- [5] KIRBY, R. Problems in low-dimensional topology. In: *Geometric Topology* (W. H. Kazez, ed.). AMS/IP Stud. Adv. Math. 2, Part 2. Amer. Math. Soc. (1997), 35–473.
- [6] LAITINEN, E. and I. MADSEN. Topological classification of  $SL_2(\mathbb{F}_p)$  space forms. In: *Algebraic Topology* (J. L. Dupont and I. H. Madsen, eds.). Lecture Notes in Math. 763 (1979), 1–27.
- [7] MADSEN, I., C. B. THOMAS and C. T. C. WALL. The topological spherical space form problem III: Dimensional bounds and smoothings. *Pacific J. Math.* 106 (1983), 135–143.
- [8] MILNOR, J. Groups which act on  $S^n$  without fixed points. *Amer. J. Math.* 79 (1957), 623–630.
- [9] PETRIE, T. Free metacyclic group actions on homotopy spheres. *Ann. of Math.* (2) 94 (1971), 108–124.
- [10] RUBINSTEIN, J. H. An algorithm to recognize the 3-sphere. In: *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*. Birkhäuser, 1995.
- [11] SUZUKI, M. On finite groups with cyclic Sylow subgroups for all odd primes. *Amer. J. Math.* 77 (1955), 657–691.
- [12] THOMAS, C. B. Almost linear actions by  $SL_2(p)$  on  $S^{2n-1}$ . *Forum Math.* 9 (1997), 751–760.
- [13] THURSTON, W. P. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)* 6 (1982), 357–381.
- [14] WALL, C. T. C. *Surgery on Compact Manifolds* (2nd edition, ed. by A. A. Ranicki). Math. Surveys Monogr. 69, Amer. Math. Soc., 1999.

- [15] WOLF, J. A. *Spaces of Constant Curvature*. McGraw-Hill, 1967.
- [16] — A contact structure for odd-dimensional spherical space forms. *Proc. Amer. Math. Soc.* 19 (1968), 196.

*(Reçu le 14 août 2000)*

Hansjörg Geiges

Mathematisch Instituut  
Universiteit Leiden  
Postbus 9512  
NL-2300 RA Leiden  
The Netherlands  
*e-mail*: geiges@math.leidenuniv.nl

Charles B. Thomas

Centre for Mathematical Sciences  
University of Cambridge  
Wilberforce Road  
GB-Cambridge CB3 0WB  
United Kingdom  
*e-mail*: C.B.Thomas@dpmms.cam.ac.uk