1. The hive model

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

SATURATION CONJECTURE. *Let* $(\lambda, \mu, \nu) \in \mathbf{Z}^{3n}$ *and* $N > 0$. *Then* $(\lambda, \mu, \nu) \in T_n$ *if and only if* $(N\lambda, N\mu, N\nu) \in T_n$.

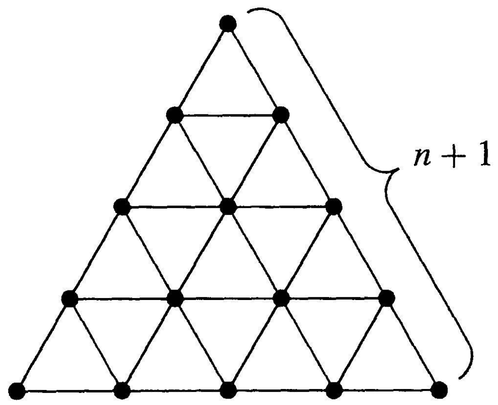
In other words T_n is saturated in $\mathbf{Z}^{3n}$. Note that the implication "only if" is a trivial consequence of the fact that T_n is a semi-group or of the original Littlewood-Richardson rule.

In July 1998, Knutson and Tao gave a proof of this conjecture, using two wonderful new constructions of polytopes, whose lattice points count Littlewood-Richardson coefficients. These constructions are called the *hive* and *honeycomb* models. Earlier Berenstein and Zelevinsky had defined equivalent polytopes, but with more complicated descriptions. In the first preprint of Knutson and Tao's paper, both hives and honeycombs were used. However, in their later version [10], hives were eliminated from the proof.

The goal of this exposition is to present a simple and complete proof using only the hive model. It is based on Knutson and Tao's first preprint, and most constructions used here come directly from this preprint. One innovation, in Section 3, is the construction of a graph from a hive, which is used to simplify their argument. In an appendix by Fulton it is shown that the hive model is equivalent to the original Littlewood-Richardson rule. We thank W. Fulton, S. Hosten, F. Sottile, and B. Sturmfels for useful discussions, and Knutson and Tao for keeping us informed about their progress. We are also grateful to the referee for many useful suggestions.

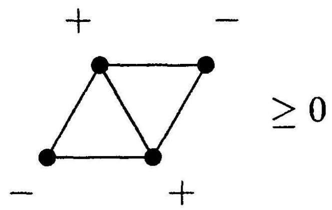
1. THE HIVE MODEL

We start with a triangular array of *hive vertices*, $n+1$ on each side:



This array is called the *(big) hive triangle*. When lines are drawn through the hive vertices as shown, the hive triangle is split up into n^2 *small triangles*. By a *rhombus* we mean the union of two small triangles next to each other.

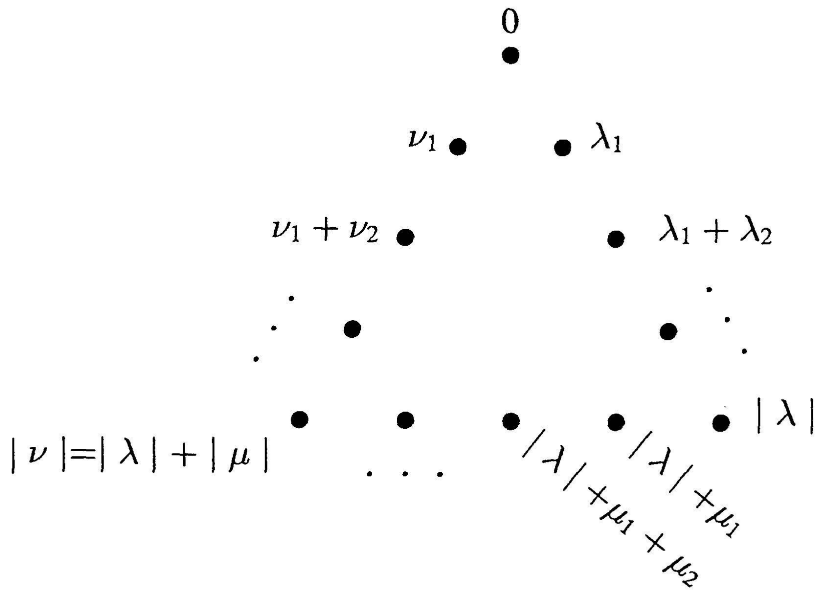
Let H be the set of hive vertices and $\mathbf{R}^H$ the labelings of these by real numbers. Each rhombus gives rise to an inequality on $\mathbf{R}^H$ saying that the sum of the labels at the obtuse vertices must be greater than or equal to the sum of the labels at the acute vertices:



A *hive* is a labeling that satisfies all rhombus inequalities. A hive is *integral* if all its labels are integers. We let $C \subset \mathbf{R}^H$ denote the convex polyhedral cone consisting of all hives.

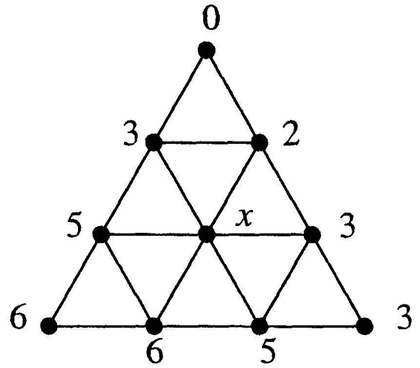
Denote by $|\lambda|$ the weight of the partition λ, which is the sum of its entries. The following theorem gives the relation between Littlewood-Richardson coefficients and hives.

THEOREM 1. *Let λ, μ, and ν be partitions with $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu}$ is the number of integral hives with border labels:*



Knutson and Tao prove this by translating hives with integer labels into tail-positive Berenstein-Zelevinsky patterns, which are known to count $c^{\nu}_{\lambda\mu}$ [1], [12]. An alternative direct proof of Fulton can be found in the appendix.

EXAMPLE 1. To compute $c_{21,21}^{321}$ we can take $n=3$ and border labels as in the picture.



Let x be the undetermined label of the middle hive vertex. Then the rhombus inequalities say that $4 \leq x \leq 5$. It follows that there are two integral hives with this border, so $c_{21,21}^{321} = 2$.

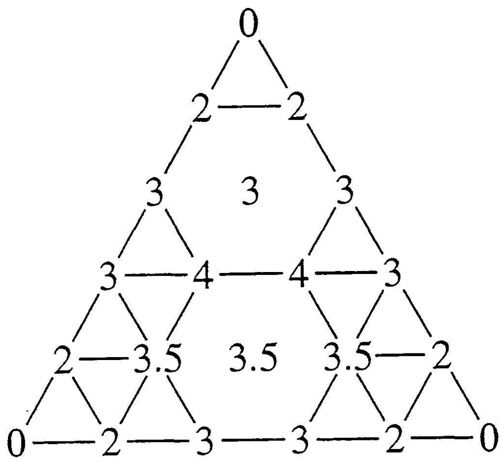
Let B be the set of border vertices, and $\rho\colon \mathbf{R}^H \to \mathbf{R}^B$ the restriction map. The restriction of a hive to the border vertices by ρ is called its *border*. For $b \in \mathbf{R}^B$, the fiber $\rho^{-1}(b) \cap C$ is easily seen to be a compact polytope, which we will call the *hive polytope* over b. If b comes from a triple of partitions as in Theorem 1, this is also called the hive polytope over the triple. We will call the vertices of a hive polytope its *corners*.

We can now describe the strategy of Knutson and Tao's proof. If $(N\lambda, N\mu, N\nu)$ is in T_n, then the hive polytope over this triple contains an integral hive. By scaling this polytope down by a factor N, it follows that the hive polytope over (λ, μ, ν) is not empty. Therefore it is enough to show that if $b \in \mathbf{Z}^B$ and $\rho^{-1}(b) \cap C \neq \varnothing$ then $\rho^{-1}(b) \cap C$ contains an integral hive.

Let ω be a functional on $\mathbf{R}^{H-B}$ which maps a hive to a linear combination of the labels at non-border vertices, with generic positive coefficients. Then for each $b \in \rho(C)$, this ω takes its maximum at a unique hive in $\rho^{-1}(b) \cap C$. The strategy is to prove that this hive is integral if b is integral.

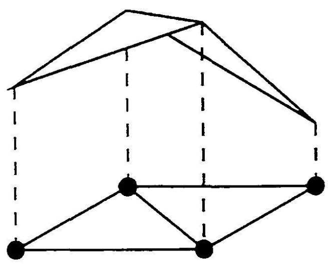
EXAMPLE 2. Even though all rhombus inequalities are integrally defined, a hive polytope over an integral border can still have non-integral corners. The following hive is an example, and therefore it does not maximize any generic positive functional ω.

In the picture we have omitted the lines across rhombi where the rhombus inequality is satisfied with equality, which makes it easy to see that this hive is a corner of its hive polytope. In fact, it is not hard to show that for $n \leq 4$ and $b \in \mathbf{Z}^B$, all corners of $\rho^{-1}(b) \cap C$ are integral hives.



2. Flatspaces

We can consider a hive as a graph over the hive triangle. At each hive vertex we use the label as the height. We then extend these heights to a graph over the entire hive triangle by using linear interpolation over each small triangle. A rhombus inequality now says that the graph over the rhombus cannot bend up across the middle line.



In this way the graph becomes the surface of a convex mountain. The graph is flat (but not necessarily horizontal) over a rhombus if and only if the rhombus inequality is satisfied with equality.

We define a *flatspace* to be a maximal connected union of small triangles such that any contained rhombus is satisfied with equality. The flatspaces split the hive triangle up in disjoint regions over which the mountain is flat. The flatspaces of the hive in Example 2 consist of two hexagons and 13 small triangles.

Flatspaces have a number of nice properties. We will list the ones we need below. Since all of these are straightforward to prove directly from the definitions, we will simply give intuitive reasons for them.