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3. In order to make use of π_1 in locating other (not locally minimal) closed geodesics without non-degeneracy condition, one has to extend [Gr] to the non simply connected situation. When V is homeomorphic to $V_0 \times S^1$ and V_0 is simply connected, we can apply [Gr] directly and get $N_m(V) \geq C \log(m)$ for some $C > 0$. (Probably this is true when $H_1(V)$ is infinite or at least when $\pi_1(V) = \mathbf{Z}$.) The last estimate can be sharpened and we show this here for the simplest example when V_0 is the sphere S^3 and the proof is obvious [Gr].

Let V be homeomorphic to $S^3 \times S^1$. Then there exist closed geodesics $g_j^i \subset V$ (not necessarily simple) such that

1. Each g_j^i , $i, j = 1, 2, \dots$, represents $\gamma^i \in \pi_1(V)$ where γ is a generator in $\pi_1(V)$.
2. For each i the geodesic g_1^i is the shortest in its homotopy class.

Denote by $|g_j^i|$ the length of g_j^i .

3. $|g_1^{i+k}| + C \geq |g_1^i| + |g_1^k| \geq |g_1^{i+k}|$, where $C \geq 0$ and $i, k = 1, 2, \dots$
4. $|g_j^i| + C \geq |g_{j+1}^i| \geq |g_j^i|$ for some $C > 0$ and $i, j = 1, 2, \dots$
5. $||g_j^i| - |g_j^k|| \leq C|i - k|$ for some $C > 0$ and $i, j, k = 1, 2, \dots$
6. $|g_j^i| \geq \frac{i}{C}$ for some $C > 0$ and $i, j = 1, 2, \dots$

COROLLARY. *If V is as above, then $\limsup_{m \rightarrow \infty} \frac{N_m(V)}{m^2} \geq \text{const} > 0$.*

All our estimates give a rather poor approximation to the (unknown) reality. Probably, in most cases N_m grows exponentially. That is so, of course, for “ C^0 -generic” manifolds (“ C^0 -generic” is used for C^0 -generic manifolds having uncountably many closed geodesics).

REFERENCES

- [Fr] FRANKS, J. Anosov diffeomorphisms. In: *Global Analysis, Berkeley, 1968*. Proc. Symp. Pure Math. 14, Amer. Math. Soc. 1970, 61–93.
- [G-M] GROMOLL, D. and W. MEYER. Periodic geodesics on compact manifolds. *J. Differential Geom.* 3 (1969), 493–510.
- [Gr] GROMOV, M. Homotopical effects of dilatation. *J. Differential Geom.* 13 (1978), 303–310.
- [KI] KLINGENBERG, W. *Lectures on Closed Geodesics*. Third ed. Mathematisches Institut der Universität Bonn, 1977.

- [Ma] MANNING, A. Topological entropy and the first homology group. *Springer Lecture Notes in Math.* 468 (1974), 185–191.
- [Nie] NIELSEN, J. Über topologische Abbildungen geschlossener Flächen. *Abh. Math. Sem. Univ. Hamburg* 3 (1924), 246–260.
- [Pu] PUGH, C. On the entropy conjecture. *Springer Lecture Notes in Math.* 468 (1974), 257–262.
- [Sh] SHUB, M. Endomorphisms of compact manifolds. *Amer. J. Math.* 91 (1969), 185–199.
- [Sh-S] SHUB, M. and D. SULLIVAN. A remark on the Lefschetz fixed point formula. *Topology* 13 (1974), 189–191.

WHY THE APPENDICES WERE NOT WRITTEN :

AUTHOR'S APOLOGIES TO THE READERS

APPENDIX 2. The stable homeomorphism suggests a geometric link between the homotopy and topological invariance of Pontryagin classes, at least for manifolds with negative curvature but I did not manage to forge this to my satisfaction till 1996 (see [Gro₂]); also see [Fa-Jo] for a deeper analysis.

APPENDIX 3. One can define a notion of hyperbolicity for an automorphism α of an arbitrary finitely generated group Γ , such that (Γ, α) functorially defines a Bowen-Franks hyperbolic system (see [Gro₁]). Unfortunately, this class of (Γ, α) is rather limited, e.g. is not closed under free products and does not include hyperbolic automorphisms of surface groups. I still do not know what the right setting is.

APPENDIX 4. An obvious example of semi-hyperbolicity is provided by non-strictly expanding endomorphisms, where the geometric picture is rather clear. However, I still do not see a functorial description, in the spirit of the symbolic dynamics, of more general semi-hyperbolic systems, not even for the geodesic (or Weil chamber) flows on locally symmetric spaces (compare [B-G-S] and [Br-Ha]).

APPENDIX 5. The section on entropy was inspired by Manning's paper [Ma₁], but I was unaware of the prior paper by Dinaburg (see [Din]) that essentially contained the entropy estimate for geodesic flows (also discussed in [Ma₂]). On the other hand, estimating the entropy of an endomorphism (or an automorphism) f in terms of $f_*: \pi_1 \rightarrow \pi_1$ appears now much less clear than it seemed to me back in 1976. It is not hard to bound the entropy from below