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Artikel: THREE REMARKS ON GEODESIC DYNAMICS AND FUNDAMENTAL GROUP

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where $h = D(g)$, $t, x \in g$, $y(t) \in h$, $y(t) = P_h \circ f_0(t)$. When c is large enough, the map F_c is a homeomorphism and it is obviously equivariant. Returning to V and W we get the geodesic homeomorphism $S(V) \to S(W)$.

GENERALIZATIONS

The hyperbolic ideas of Morse were successfully applied to discrete type systems by Shub (*expanding endomorphisms*, see [Sh]) and Franks (π_1-*diffeomorphisms*, see [Fr]). Their results are discussed (and slightly generalized) in Appendix 3.

From a global geometric point of view generalizations of totally hyperbolic systems must include manifolds of nonpositive curvature and correspondingly semihyperbolic systems. (See Appendix 4.)

In differential dynamics most attention has always been paid to "local" versions of hyperbolicity (stability, Anosov's systems, Axiom A diffeomorphisms of Smale). We do not touch here upon that more analytical line of development of Morse's ideas.

§2. ENTROPY

Take a closed Riemannian manifold V, consider its universal covering X and denote by $\mathrm{Vol}_x(R)$, $x \in X$, the volume of the ball of radius R centered at x. Set $H(V) = \lim_{R\to\infty} \log \mathrm{Vol}_x(R)$. The limit obviously exists and does not depend on x. Denote by $h(V)$ the topological entropy of the geodesic flow in $S(V)$.

ENTROPY ESTIMATE. *We have* $h(V) \geq H(V)$.

COROLLARY. *If the fundamental group* $\pi_1(V)$ *can be presented by* k *generators and one relation and* $\mathrm{Diam}(V) \leq 1$ (Diam means the *diameter* of V), *then* $h(V) \geq \log(k-1)$.

The entropy estimate immediately follows from the Covering lemma.

COVERING LEMMA. *Take a compact manifold S and consider a regular (normal) covering $T \to S$ with the action of $\Gamma = \pi_1(S)/\pi_1(T)$. Fix a fundamental domain $D \subset T$ and denote by $N(U)$, $U \subset X$, the number of motions $\gamma \in \Gamma$ such that the intersection $\gamma(U) \cap D$ is not empty. Consider an action of the group $\mathbf{R}$ of reals in S and its lifting to T.*

The entropy h of the action of $\mathbf{R}$ in S satisfies

$$h \geq \liminf_{r \to \infty} \frac{1}{|r|} \log N(r(D)),$$

where $r(D)$ denotes the image of D under the lifted action of $r \in \mathbf{R}$ in T.

Proof. Use the definition of entropy involving coverings.

This lemma (and the proof) holds for discrete time systems and immediately implies Manning's estimate of the topological entropy of an $f \colon S \to S$ in terms of the spectral radius of $f_* : H_1(S;\mathbf{R}) \to H_1(S;\mathbf{R})$. See [Ma], [Pu]. In Appendix 5 we show how to make use of the whole group $\pi_1(S)$.

§3. PERIODIC ORBITS

For maps $f \colon S \to S$ there are several ways to estimate from below the number $\operatorname{card}(\operatorname{Fix}(f^m))$ of all points of period m. Denote by $L(f)$ the Lefschetz number $\sum_{i=0}^{i=n}(-1)^i \operatorname{Trace}(f_{*i})$, where $i = \dim S$ and $f_{*i} \colon H_i(S;\mathbf{R}) \to H_i(S;\mathbf{R})$.

(***L***) *If all periodic points are nondegenerate (say, f is smooth and generic), then* $\operatorname{card}(\operatorname{Fix}(f^m)) \geq |L(f)|$ (Lefschetz).

(***Sh-S***) *If f is smooth and $\lim_{m\to\infty} |L(f^m)| = \infty$, then*

$$\lim_{m\to\infty} \operatorname{card}(\operatorname{Fix}(f^m)) = \infty$$

(Shub and Sullivan, see [Sh-S]).

(***Nie***) *Generally there is no way to extend the* (***L***)*-estimate to all maps, but in the presence of the fundamental group one can apply the Nielsen theory of fixed-point classes* (see [Nie] and Appendix 6). *This theory yields in many cases the estimate*

$$\operatorname{card}(\operatorname{Fix}(F)) \geq const\, |L(f)|,$$

and sometimes even $\operatorname{card}(\operatorname{Fix}(f^m)) \geq |L(F^m)|$, *where f is an arbitrary continuous map.*