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We can restate assertion (b) of Theorem 7.1 as follows.

**THEOREM 7.2.** *Let  $A$  be an associative ring with involution, in which 2 is invertible. Assume that  $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$ . Then there exists a natural homomorphism  $Res$  such that the sequence*

$$0 \rightarrow W(A) \rightarrow W(A[t, t^{-1}]) \xrightarrow{Res} W(A) \rightarrow 0$$

*is split exact. The homomorphism  $Res$  restricts to an isomorphism of  $t \cdot W(A)$  onto  $W(A)$ .*

### 8. TWO COUNTEREXAMPLES

In this section we show that the map  $W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$ , in general, is neither surjective nor injective.

**EXAMPLE 8.1.** We first recall the Mayer-Vietoris sequence associated to a cartesian square of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ f \downarrow & & \downarrow g \\ \bar{R} & \longrightarrow & \bar{S} \end{array}$$

be a cartesian diagram of commutative rings, with  $f$  or  $g$  surjective. Denote by  $\widetilde{K}_0$  the kernel of the rank function on  $K_0$ . Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\bar{R}) \times K_1(S) & \longrightarrow & K_1(\bar{S}) & \longrightarrow & \widetilde{K}_0(R) & \longrightarrow & \widetilde{K}_0(\bar{R}) \times \widetilde{K}_0(S) & \longrightarrow & \widetilde{K}_0(\bar{S}) \\ \downarrow \det & & \downarrow \det & & \downarrow \wedge^{\max} & & \downarrow \wedge^{\max} & & \downarrow \wedge^{\max} \\ \mathbf{G}_m(\bar{R}) \times \mathbf{G}_m(S) & \longrightarrow & \mathbf{G}_m(\bar{S}) & \longrightarrow & \text{Pic}(R) & \longrightarrow & \text{Pic}(\bar{R}) \times \text{Pic}(S) & \longrightarrow & \text{Pic}(\bar{S}) \end{array}$$

Let  $A$  be the local ring at the origin of the complex plane curve  $Y^2 = X^2 - X^3$ ,  $\widetilde{A}$  the normalisation of  $A$  and  $\mathfrak{c}$  the conductor of  $\widetilde{A}$  in  $A$ . Applying the big diagram above to the cartesian squares

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ (A/\mathfrak{c}) & \longrightarrow & (\widetilde{A}/\mathfrak{c}) \end{array} \quad \text{and} \quad \begin{array}{ccc} A[t, t^{-1}] & \longrightarrow & \widetilde{A}[t, t^{-1}] \\ \downarrow & & \downarrow \\ (A/\mathfrak{c})[t, t^{-1}] & \longrightarrow & (\widetilde{A}/\mathfrak{c})[t, t^{-1}] \end{array}$$

it is easy to see that  $\widetilde{K}_0(A[t, t^{-1}]) = \mathbf{C}^* \oplus \mathbf{Z} = \text{Pic}(A[t, t^{-1}])$ . This shows that a projective  $A[t, t^{-1}]$ -module  $P$  is stably free if and only if its maximal exterior power  $\bigwedge^{\max}(P)$  is isomorphic to  $A[t, t^{-1}]$ .

Let  $I$  be an ideal representing  $(1, 1)$  in  $\mathbf{C}^* \oplus \mathbf{Z} = \text{Pic}(A[t, t^{-1}])$ . The module underlying the space  $H(I \oplus A[t, t^{-1}] \oplus A[t, t^{-1}])$  is free. In fact it is stably free because its determinant is trivial, hence, by a well-known cancellation theorem it is free. This shows that  $H(I \oplus A[t, t^{-1}] \oplus A[t, t^{-1}])$  is a quadratic space of the form  $(P_0[t, t^{-1}], \alpha)$  with  $P_0$  free of rank 6 over  $A$ . Clearly this space represents the zero element of  $W(A[t, t^{-1}])$ . We claim that its class in  $W'(A[t, t^{-1}])$  is not trivial.

Since  $A$  is local, projective modules extended from  $A$  are free. If  $H(I \oplus A[t, t^{-1}] \oplus A[t, t^{-1}])$  were hyperbolic in  $W'(A[t, t^{-1}])$  it would be stably isometric to  $H(A[t, t^{-1}] \oplus A[t, t^{-1}] \oplus A[t, t^{-1}])$  and hence, by the quadratic cancellation theorem (see [4], VI, 6.2.5), it would be isometric to it. Recall that, for any commutative ring  $R$  in which 2 is invertible and any finitely generated projective  $R$ -module  $P$ , the even Clifford algebra  $C_0$  of  $H(P)$  is of the form

$$C_0 = \text{End}_R(\bigwedge^{\text{even}}(P)) \times \text{End}_R(\bigwedge^{\text{odd}}(P)),$$

where  $\bigwedge^{\text{even}}(P)$  (respectively  $\bigwedge^{\text{odd}}(P)$ ) is the even (respectively odd) part of the exterior algebra of  $P$ . In the case  $P = I \oplus A[t, t^{-1}] \oplus A[t, t^{-1}]$  we have

$$C_0 = \text{End}_{A[t, t^{-1}]}(A[t, t^{-1}]^2 \oplus I^2) \times \text{End}_{A[t, t^{-1}]}(A[t, t^{-1}]^2 \oplus I^2).$$

Suppose now that  $H(I \oplus A[t, t^{-1}]^2)$  and  $H(A[t, t^{-1}]^3)$  are isometric. In this case their even Clifford algebras would be isomorphic, hence the algebra  $\text{End}_{A[t, t^{-1}]}(A[t, t^{-1}]^2 \oplus I^2)$  would be a  $4 \times 4$  matrix algebra. By Morita theory the module  $A[t, t^{-1}]^2 \oplus I^2$  would be of the form  $J^4$  for some invertible ideal  $J$ . Taking the fourth exterior power of both sides we would have  $I^2 = J^4$ , which is impossible because  $I$  represents  $(1, 1)$  in  $\mathbf{C}^* \oplus \mathbf{Z}$ .

This shows that, even for a one-dimensional local domain, the map  $W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$  may fail to be injective.

EXAMPLE 8.2. We define a commutative ring  $A$  by the cartesian diagram of real algebras

$$(1) \quad \begin{array}{ccc} A & \longrightarrow & \mathbf{R}[X, Y] \\ \downarrow & & \downarrow \pi \\ \mathbf{R} & \xrightarrow{\iota} & C \end{array}$$

where  $C = \mathbf{R}[x, y] = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$ ,  $\pi$  is the canonical projection and  $\iota$  the canonical injection. Then  $C \oplus C$  is the direct sum of its two submodules

$$P = C_{\frac{1}{2}}(y + 1, -x) + C_{\frac{1}{2}}(-x, 1 - y) \quad \text{and} \quad P' = C_{\frac{1}{2}}(1 - y, x) + C_{\frac{1}{2}}(x, 1 + y)$$

and we can define an automorphism  $\alpha$  of  $C[t, t^{-1}] \oplus C[t, t^{-1}]$  as the identity on  $P'$  and multiplication by  $t$  on  $P$ . With respect to the canonical basis of  $C[t, t^{-1}] \oplus C[t, t^{-1}]$ ,

$$\alpha = \frac{1}{2} \begin{pmatrix} t(1 + y) + 1 - y & -tx + x \\ -tx + x & t(1 - y) + 1 + y \end{pmatrix}.$$

The matrix  $\alpha$  has determinant equal to  $t$  and thus lies in  $\text{GL}_2(C[t, t^{-1}])$ . According to Theorem 7.4 of [1] its class in  $K_1(C[t, t^{-1}])$  is the image of  $P$  by the canonical injection  $\lambda$  mentioned in §2. It is easy to see that  $P$  is not free over  $C$ . In fact it turns out to represent the non trivial class of  $\text{Pic}(C) = \mathbf{Z}/2$ . Since the homomorphism  $\iota$  in the cartesian square that defines  $A$  is surjective, tensoring the diagram with  $\mathbf{R}[t, t^{-1}]$  yields a Milnor patching diagram

$$\begin{array}{ccc} A[t, t^{-1}] & \longrightarrow & \mathbf{R}[X, Y][t, t^{-1}] \\ \downarrow & & \downarrow \pi \\ \mathbf{R}[t, t^{-1}] & \xrightarrow{\iota} & C[t, t^{-1}] \end{array}$$

We can use this diagram and the matrix  $\alpha$  (see for instance [1], Chapter IX, Theorem 5.1) to patch a rank 2 free module  $Q$  over  $\mathbf{R}[X, Y][t, t^{-1}]$  with a rank 2 free module  $R$  over  $\mathbf{R}[t, t^{-1}]$  and get a rank 2 projective module

$$M = \{(q, r) \in Q \times R \mid \alpha(\pi_*(q)) = \iota_*(r)\}$$

over  $A[t, t^{-1}]$ . We now equip  $M$  with a skew-symmetric structure. To do this we put on  $Q$  and on  $R$  the skew-symmetric structures defined, respectively, by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}.$$

Since  $\alpha^* \tau \alpha = \sigma$ , the skew-symmetric structures  $\sigma: Q \rightarrow Q^*$  and  $\tau: R \rightarrow R^*$  are compatible with the patching and therefore they define a skew-symmetric structure  $\varphi: M \rightarrow M^*$  on  $M$ .

We claim that the class of this space is not in the image of  $W'([t, t^{-1}])$ . Extending to  $K_{-1}$  the Mayer-Vietoris sequence associated to (1) (see [1], Chapter XII, Theorem 8.3) we get an exact sequence

$$K_0(\mathbf{R}[X, Y]) \oplus K_0(\mathbf{R}) \rightarrow K_0(C) \rightarrow K_{-1}(A) \rightarrow K_{-1}(\mathbf{R}[X, Y]) \oplus K_{-1}(\mathbf{R}).$$

From the fact that regular rings have a vanishing  $K_{-1}$ , that  $K_0(\mathbf{R}[X, Y]) = K_0(\mathbf{R}) = \mathbf{Z}$  and that  $K_0(C) = \mathbf{Z} \oplus \mathbf{Z}/2$ , where the element of order 2 is the class of  $P$ , we easily deduce that  $K_{-1}(A) = \mathbf{Z}/2$ , generated by the image of  $M$ . Thus, by Corollary 2.4, the class of  $M$  generates  $H^2(\mathbf{Z}/2, K_0(A[t, t^{-1}])/K_0(A)) = \mathbf{Z}/2$ . Consider now the homomorphism

$$\omega: W(A[t, t^{-1}]) \longrightarrow H^2(\mathbf{Z}/2, K_0(A[t, t^{-1}])/K_0(A))$$

obtained by associating to any space its underlying projective module. Since  $\omega((M, \varphi)) \neq 0$ ,  $(M, \varphi)$  cannot be Witt equivalent to a space supported by a module extended from  $A$ . This shows that the map  $W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$  is not surjective.

REMARK 8.3. We suspect that even if the assumption of (a) is satisfied the map  $W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$  may not be injective, but we did not find an example to confirm our suspicion.

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