Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 46 (2000)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE WITT GROUP OF LAURENT POLYNOMIALS

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Kapitel: 8. TWO COUNTEREXAMPLES

DOI: https://doi.org/10.5169/seals-64807

We can restate assertion (b) of Theorem 7.1 as follows.

THEOREM 7.2. *Let A be an associative ring with involution, in which 2 is invertible. Assume that $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$. Then there exists a natural homomorphism Res such that the sequence*

$$0 \longrightarrow W(A) \longrightarrow W(A[t,t^{-1}]) \xrightarrow{Res} W(A) \longrightarrow 0$$

is split exact. The homomorphism Res restricts to an isomorphism of $t \cdot W(A)$ onto $W(A)$.

8. TWO COUNTEREXAMPLES

In this section we show that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$, in general, is neither surjective nor injective.

EXAMPLE 8.1. We first recall the Mayer-Vietoris sequence associated to a cartesian square of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \bar{R} & \longrightarrow & \bar{S} \end{array}$$

be a cartesian diagram of commutative rings, with f or g surjective. Denote by $\widetilde{K_0}$ the kernel of the rank function on K_0. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\bar{R}) \times K_1(S) & \longrightarrow & K_1(\bar{S}) & \longrightarrow & \widetilde{K_0}(R) & \longrightarrow & \widetilde{K_0}(\bar{R}) \times \widetilde{K_0}(S) & \longrightarrow & \widetilde{K_0}(\bar{S}) \\ \big\downarrow{\scriptstyle \det} & & \big\downarrow{\scriptstyle \det} & & \big\downarrow{\scriptstyle \wedge^{\max}} & & \big\downarrow{\scriptstyle \wedge^{\max}} & & \big\downarrow{\scriptstyle \wedge^{\max}} \\ \mathbf{G}_m(\bar{R}) \times \mathbf{G}_m(S) & \longrightarrow & \mathbf{G}_m(\bar{S}) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}(\bar{S}) \end{array}$$

Let A be the local ring at the origin of the complex plane curve $Y^2 = X^2 - X^3$, $\widetilde{A}$ the normalisation of A and $\mathfrak{c}$ the conductor of $\widetilde{A}$ in A. Applying the big diagram above to the cartesian squares

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{A} \\ \big\downarrow & & \big\downarrow \\ (A/\mathfrak{c}) & \longrightarrow & (\widetilde{A}/\mathfrak{c}) \end{array} \quad \text{and} \quad \begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \widetilde{A}[t,t^{-1}] \\ \big\downarrow & & \big\downarrow \\ (A/\mathfrak{c})[t,t^{-1}] & \longrightarrow & (\widetilde{A}/\mathfrak{c})[t,t^{-1}] \end{array}$$

it is easy to see that $\widetilde{K_0}(A[t,t^{-1}]) = \mathbf{C}^* \oplus \mathbf{Z} = \mathrm{Pic}(A[t,t^{-1}])$. This shows that a projective $A[t,t^{-1}]$-module P is stably free if and only if its maximal exterior power $\bigwedge^{\max}(P)$ is isomorphic to $A[t,t^{-1}]$.

Let I be an ideal representing $(1,1)$ in $\mathbf{C}^* \oplus \mathbf{Z} = \mathrm{Pic}(A[t,t^{-1}])$. The module underlying the space $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is free. In fact it is stably free because its determinant is trivial, hence, by a well-known cancellation theorem it is free. This shows that $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is a quadratic space of the form $(P_0[t,t^{-1}],\alpha)$ with P_0 free of rank 6 over A. Clearly this space represents the zero element of $W(A[t,t^{-1}])$. We claim that its class in $W'(A[t,t^{-1}])$ is not trivial.

Since A is local, projective modules extended from A are free. If $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ were hyperbolic in $W'(A[t,t^{-1}])$ it would be stably isometric to $H(A[t,t^{-1}] \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ and hence, by the quadratic cancellation theorem (see [4], VI, 6.2.5), it would be isometric to it. Recall that, for any commutative ring R in which 2 is invertible and any finitely generated projective R-module P, the even Clifford algebra C_0 of $H(P)$ is of the form

$$C_0 = \mathrm{End}_R(\textstyle\bigwedge^{even}(P)) \times \mathrm{End}_R(\textstyle\bigwedge^{odd}(P)),$$

where $\bigwedge^{even}(P)$ (respectively $\bigwedge^{odd}(P)$) is the even (respectively odd) part of the exterior algebra of P. In the case $P = I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}]$ we have

$$C_0 = \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2) \times \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2).$$

Suppose now that $H(I \oplus A[t,t^{-1}]^2)$ and $H(A[t,t^{-1}]^3)$ are isometric. In this case their even Clifford algebras would be isomorphic, hence the algebra $\mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2)$ would be a 4×4 matrix algebra. By Morita theory the module $A[t,t^{-1}]^2 \oplus I^2$ would be of the form J^4 for some invertible ideal J. Taking the fourth exterior power of both sides we would have $I^2 = J^4$, which is impossible because I represents $(1,1)$ in $\mathbf{C}^* \oplus \mathbf{Z}$.

This shows that, even for a one-dimensional local domain, the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may fail to be injective.

EXAMPLE 8.2. We define a commutative ring A by the cartesian diagram of real algebras

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{R}[X,Y] \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{R} & \xrightarrow{\ \iota\ } & C \end{array} \qquad (1)$$

where $C = \mathbf{R}[x,y] = \mathbf{R}[X,Y]/(X^2+Y^2-1)$, π is the canonical projection and ι the canonical injection. Then $C \oplus C$ is the direct sum of its two submodules

$$P = C\tfrac{1}{2}(y+1,-x) + C\tfrac{1}{2}(-x,1-y) \quad \text{and} \quad P' = C\tfrac{1}{2}(1-y,x) + C\tfrac{1}{2}(x,1+y)$$

and we can define an automorphism α of $C[t,t^{-1}] \oplus C[t,t^{-1}]$ as the identity on P' and multiplication by t on P. With respect to the canonical basis of $C[t,t^{-1}] \oplus C[t,t^{-1}]$,

$$\alpha = \frac{1}{2}\begin{pmatrix} t(1+y)+1-y & -tx+x \\ -tx+x & t(1-y)+1+y \end{pmatrix}.$$

The matrix α has determinant equal to t and thus lies in $\mathrm{GL}_2(\mathrm{C}[\mathrm{t},\mathrm{t}^{-1}])$. According to Theorem 7.4 of [1] its class in $K_1(C[t,t^{-1}])$ is the image of P by the canonical injection λ mentioned in §2. It is easy to see that P is not free over C. In fact it turns out to represent the non trivial class of $\mathrm{Pic}(C) = \mathbf{Z}/2$. Since the homomorphism ι in the cartesian square that defines A is surjective, tensoring the diagram with $\mathbf{R}[t,t^{-1}]$ yields a Milnor patching diagram

$$\begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \mathbf{R}[X,Y][t,t^{-1}] \\ \downarrow & & \downarrow \pi \\ \mathbf{R}[t,t^{-1}] & \xrightarrow{\ \iota\ } & C[t,t^{-1}] \end{array}$$

We can use this diagram and the matrix α (see for instance [1], Chapter IX, Theorem 5.1) to patch a rank 2 free module Q over $\mathbf{R}[X,Y][t,t^{-1}]$ with a rank 2 free module R over $\mathbf{R}[t,t^{-1}]$ and get a rank 2 projective module

$$M = \{(q,r) \in Q \times R \mid \alpha(\pi_*(q)) = \iota_*(r)\}$$

over $A[t,t^{-1}]$. We now equip M with a skew-symmetric structure. To do this we put on Q and on R the skew-symmetric structures defined, respectively, by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}.$$

Since $\alpha^*\tau\alpha = \sigma$, the skew-symmetric structures $\sigma\colon Q \to Q^*$ and $\tau\colon R \to R^*$ are compatible with the patching and therefore they define a skew-symmetric structure $\varphi\colon M \to M^*$ on M.

We claim that the class of this space is not in the image of $W'([t,t^{-1}])$. Extending to K_{-1} the Mayer-Vietoris sequence associated to (1) (see [1], Chapter XII, Theorem 8.3) we get an exact sequence

$$K_0(\mathbf{R}[X,Y]) \oplus K_0(\mathbf{R}) \to K_0(C) \to K_{-1}(A) \to K_{-1}(\mathbf{R}[X,Y]) \oplus K_{-1}(\mathbf{R}).$$

From the fact that regular rings have a vanishing K_{-1}, that $K_0(\mathbf{R}[X,Y]) = K_0(\mathbf{R}) = \mathbf{Z}$ and that $K_0(C) = \mathbf{Z} \oplus \mathbf{Z}/2$, where the element of order 2 is the class of P, we easily deduce that $K_{-1}(A) = \mathbf{Z}/2$, generated by the image of M. Thus, by Corollary 2.4, the class of M generates $H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = \mathbf{Z}/2$. Consider now the homomorphism

$$\omega\colon W(A[t,t^{-1}]) \longrightarrow H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A))$$

obtained by associating to any space its underlying projective module. Since $\omega((M,\varphi)) \neq 0$, (M,φ) cannot be Witt equivalent to a space supported by a module extended from A. This shows that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ is not surjective.

REMARK 8.3. We suspect that even if the assumption of (a) is satisfied the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may not be injective, but we did not find an example to confirm our suspicion.

ACKNOWLEDGMENT. We warmly thank Paul Balmer for carefully reading various versions of this paper, dramatically reducing our output of mistakes.

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(Reçu le 23 mars 2000)

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