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PROPOSITION 6.8. *Let A be a commutative semilocal ring in which 2 is invertible. Let (P, α) be a quadratic space over A. If (P, α) is isometric to $(P, t\cdot\alpha)$ over $A[t, t^{-1}]$, then (P, α) is hyperbolic.*

Proof. Let ξ be the class of (P, α) in $W(A)$. In $W'(A[t])$ we have $\xi = t\cdot\xi$. Applying *Res* to both sides we obtain $\xi = 0$. Since A is semilocal, by Witt's cancelletion theorem we conclude that (P, α) is hyperbolic. □

7. THE WITT GROUP OF LAURENT POLYNOMIALS

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 7.1. *Let A be an associative ring with involution in which 2 is invertible. Let*

$$\varphi\colon W'(A[t, t^{-1}]) \to W(A[t, t^{-1}])$$

be the canonical homomorphism.

(a) *If $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$, then φ is surjective.*

(b) *If $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$, then φ is an isomorphism.*

Proof of (a). Corollary 2.4 implies that

$$H^2(\mathbf{Z}/2, K_0(A[t, t^{-1}])/K_0(A)) = 0\,.$$

This means that every projective $A[t, t^{-1}]$-module P is in the same class as some projective module of the form

$$P_0[t, t^{-1}] \oplus Q \oplus Q^*\,,$$

where P_0 is a projective A-module. Therefore, adding to a space (P, α) a hyperbolic space $H(Q')$ with $Q \oplus Q'$ free, we may assume that P is of the form $P_0[t, t^{-1}]$. This means precisely that the class of (P, α) is in the image of $W'(A[t, t^{-1}])$. □

Proof of (b). Surjectivity is obvious, because by assumption every projective $A[t, t^{-1}]$-module is stably extended from A. Suppose that the class of a space $(P_0[t, t^{-1}], \alpha)$ vanishes in $W(A[t, t^{-1}])$. This means that, for some Q and R, there exists an isometry

$$(P_0[t, t^{-1}], \alpha) \perp H(Q) \simeq H(R)\,.$$

Adding to both sides a suitable $H(A[t, t^{-1}]^n)$ we may replace Q and R by extended modules $Q_0[t, t^{-1}]$ and $R_0[t, t^{-1}]$. Then the isometry means precisely that the class of $(P_0[t, t^{-1}], \alpha)$ vanishes in $W'(A[t, t^{-1}])$. □

We can restate assertion (b) of Theorem 7.1 as follows.

THEOREM 7.2. *Let A be an associative ring with involution, in which 2 is invertible. Assume that $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$. Then there exists a natural homomorphism Res such that the sequence*

$$0 \longrightarrow W(A) \longrightarrow W(A[t,t^{-1}]) \xrightarrow{Res} W(A) \longrightarrow 0$$

is split exact. The homomorphism Res restricts to an isomorphism of $t \cdot W(A)$ onto $W(A)$.

8. TWO COUNTEREXAMPLES

In this section we show that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$, in general, is neither surjective nor injective.

EXAMPLE 8.1. We first recall the Mayer-Vietoris sequence associated to a cartesian square of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \bar{R} & \longrightarrow & \bar{S} \end{array}$$

be a cartesian diagram of commutative rings, with f or g surjective. Denote by $\widetilde{K_0}$ the kernel of the rank function on K_0. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\bar{R}) \times K_1(S) & \longrightarrow & K_1(\bar{S}) & \longrightarrow & \widetilde{K_0}(R) & \longrightarrow & \widetilde{K_0}(\bar{R}) \times \widetilde{K_0}(S) & \longrightarrow & \widetilde{K_0}(\bar{S}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \wedge^{\max}} & & \downarrow{\scriptstyle \wedge^{\max}} & & \downarrow{\scriptstyle \wedge^{\max}} \\ \mathbf{G}_m(\bar{R}) \times \mathbf{G}_m(S) & \longrightarrow & \mathbf{G}_m(\bar{S}) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}(\bar{S}) \end{array}$$

Let A be the local ring at the origin of the complex plane curve $Y^2 = X^2 - X^3$, $\widetilde{A}$ the normalisation of A and $\mathfrak{c}$ the conductor of $\widetilde{A}$ in A. Applying the big diagram above to the cartesian squares

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ (A/\mathfrak{c}) & \longrightarrow & (\widetilde{A}/\mathfrak{c}) \end{array} \quad \text{and} \quad \begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \widetilde{A}[t,t^{-1}] \\ \downarrow & & \downarrow \\ (A/\mathfrak{c})[t,t^{-1}] & \longrightarrow & (\widetilde{A}/\mathfrak{c})[t,t^{-1}] \end{array}$$