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Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ -hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon \sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon \sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N - 1$ and after N - 1 such transformations we get a linear matrix. \Box

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1 + \nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1 + \nu t)^{-1/2} = \sum {\binom{-1/2}{k} (\nu t)^k}$$

is a polynomial. From $\alpha = \epsilon \alpha^*$ we get $\alpha_0^* = \epsilon \alpha_0$ and $\nu^* \alpha_0^* = \epsilon \alpha_0 \nu$. This implies that $\tau^* \alpha_0^* = \epsilon \alpha_0 \tau$ and therefore

$$\tau^* \alpha \tau = \tau^* \alpha_0 (1 + \nu t) \tau = \alpha_0 \tau (1 + \nu t) \tau = \alpha_0$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic.

4. THE WITT GROUP OF TORSION MODULES

Let M be a finitely generated right A[t]-module and suppose that it is a *t*-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by M^{\sharp} the left A[t]-module $\operatorname{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ and we consider it as a right module through the involution on A[t].

Recall that, as an A-module, the quotient $A[t, t^{-1}]/A[t]$ can be written as a direct sum

$$A[t,t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots$$

Thus, to any $f \in \text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1}: M \to A$, which is defined as the composite of f with the projection onto At^{-1} .

PROPOSITION 4.1. The map

$$\partial = \partial_M \colon M^{\sharp} = \operatorname{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \longrightarrow \operatorname{Hom}_A(M, A) = M^*$$

obtained by associating f_{-1} to f is a functorial A-linear isomorphism.

Proof. It is clear that ∂ is A-linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define \tilde{g} by the (finite !) sum

$$\widetilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \cdots$$

It is easy to check that $\tilde{g} \in M^{\sharp}$, $(\tilde{g})_{-1} = g$ and $\tilde{f_{-1}} = f$. Functoriality is clear.

COROLLARY 4.2. For any finitely generated t-torsion module M which is projective as an A-module the canonical homomorphism $M \to M^{\sharp\sharp}$ is an isomorphism.

Proof. It suffices to remark that the diagram



commutes and that $M \xrightarrow{can} M^{**}$ is an isomorphism.

An ϵ -hermitian t-torsion space (or, briefly, a t-torsion space) is a pair (M, \langle , \rangle) consisting of a finitely generated t-torsion right A[t]-module M which is projective as an A-module, and a perfect ϵ -hermitian pairing $\langle , \rangle \colon M \times M \to A[t, t^{-1}]/A[t]$. Giving \langle , \rangle is the same, of course, as giving its adjoint $\varphi \colon M \to M^{\sharp}$ defined by $\varphi(a)(b) = \langle a, b \rangle$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

$$X^{\perp} = \{ y \in M \mid \langle x, y \rangle = 0 \quad \forall x \in X \}.$$

A sublagrangian of (M, φ) is an A[t]-submodule L of M which satisfies the following two conditions:

- (1) It is contained in its own orthogonal: $L \subseteq L^{\perp}$.
- (2) The quotient M/L is projective over A (which is the same as saying that L, as an A-module, is a direct factor of M).

A sublagrangian L is a lagrangian if $L = L^{\perp}$. A t-torsion space is *metabolic* if it has a lagrangian. The Witt group of t-torsion spaces is the quotient of the Grothendieck group of t-torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

LEMMA 4.3. Let M and N be finitely generated t-torsion modules and $i: N \to M$ an A[t]-linear homomorphism. Assume that as A-modules Mand N are projective. Then the map $i^{\sharp}: M^{\sharp} \to N^{\sharp}$ is surjective (respectively injective) if and only if $i^*: M^* \to N^*$ is surjective (respectively injective).

Proof. Look:



PROPOSITION 4.4. Let (M, φ) be a t-torsion space and L an A[t]-submodule of M. If M/L is projective over A, then $L = L^{\perp \perp}$ and L^{\perp} is a direct factor of M as an A-module.

Proof. First observe that as an A-module L is finitely generated and projective. Let $i: L \to M$ be the natural injection. By Lemma 4.3 the map $i^{\sharp} \circ \varphi$ is surjective, thus the sequence

$$0 \longrightarrow L^{\perp} \xrightarrow{j} M \xrightarrow{i^{\sharp} \circ \varphi} L^{\sharp} \longrightarrow 0$$

is exact. Hence L^{\perp} is a direct factor of M as an A-module; in particular it is A-projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{j^{\sharp} \circ \varphi^{\sharp}} (L^{\perp})^{\sharp} \longrightarrow 0.$$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$0 \longrightarrow L^{\perp \perp} \longrightarrow M \xrightarrow{j^{\sharp} \circ \varphi} (L^{\perp})^{\sharp} \longrightarrow 0$$

is exact because L^{\perp} is a direct factor of M as an A-module. Since $\varphi^{\sharp} = \pm \varphi$, comparing the last two sequences we get the result.

 \square

We now prove a fundamental result on the equivalence of t-torsion spaces.

THEOREM 4.5. Let (M, φ) be an ϵ -hermitian t-torsion space and L a sublagrangian of (M, φ) . The quotient L^{\perp}/L carries a natural structure of t-torsion ϵ -hermitian space and its class in $W_{tors}(A[t])$ is the same as that of (M, φ) .

Proof. We first prove the following lemma.

LEMMA 4.6. Let (M, φ) be any ϵ -hermitian t-torsion space. The space $(M, \varphi) \perp (M, -\varphi)$ is metabolic.

Proof of Lemma 4.6. We show that the image $L = \Delta(M)$ of the diagonal map $M \xrightarrow{\Delta} M \oplus M$ is a lagrangian. The condition $L \subseteq L^{\perp}$ is immediately verified. The quotient $(M \oplus M)/L$ is isomorphic to M, hence it is projective over A. It remains to see that $L^{\perp} \subseteq L$. If $(a,b) \in L^{\perp}$ we have $0 = \langle (a,b), (x,x) \rangle = \langle a-b,x \rangle$ for any $x \in M$. Since the pairing \langle , \rangle is perfect, this implies a = b, i.e. $(a,b) \in L$. \Box

We now prove the theorem. By Proposition 4.4, L^{\perp} is a direct factor of M as an A-module. Since $L \subseteq L^{\perp}$ is also a direct factor of M, the quotient L^{\perp}/L is projective. Denoting by $\overline{a}, \overline{b}$ the classes modulo L of two elements $a, b \in L$, we define the hermitian structure of L^{\perp}/L by $\langle \overline{a}, \overline{b} \rangle = \langle a, b \rangle$. It is clear that $\langle a, b \rangle$ only depends on \overline{a} and \overline{b} . We first check that this pairing defines a t-torsion space. It is clearly ϵ -hermitian. The injectivity of the adjoint map $L^{\perp}/L \to (L^{\perp}/L)^{\sharp}$ follows immediately from Proposition 4.4. To show surjectivity consider any A[t]-linear map $f: L^{\perp} \to A[t, t^{-1}]/A[t]$. Since L^{\perp} is a direct factor of M as an A-module, f, by Lemma 4.3, extends to an A[t]-linear map $\widetilde{f}: M \to A[t, t^{-1}]/A[t]$. Choose an $m \in M$ for which $\widetilde{f} = \langle m, \cdot \rangle$. If \widetilde{f} vanishes on L, then m is in L^{\perp} . This proves that L^{\perp}/L is a t-torsion space.

To show that L^{\perp}/L is equivalent to (M, φ) we check that the image of the diagonal map $\Delta: L^{\perp} \to M \oplus L^{\perp}/L$ is a lagrangian of $(M, -\varphi) \perp L^{\perp}/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^{\perp})$ is contained in its own orthogonal. Conversely, if $(a, \overline{b}) \in M \oplus L^{\perp}/L$ is orthogonal to every (x, \overline{x}) , then $\langle a - b, x \rangle = 0$ for every $x \in L^{\perp}$. This means that a - b is in $L^{\perp \perp}$, which by Proposition 4.4 coincides with L. We thus have $(a, \overline{b}) = (a, \overline{a}) \in \Delta(L^{\perp})$.

The next proposition connects the Witt group of t-torsion spaces with the Witt group of A.

PROPOSITION 4.7. The isomorphisms

$$\partial_M \colon \operatorname{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \to \operatorname{Hom}_A(M, A)$$

induce a surjective homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A)$$
.

Proof. Associating to any *t*-torsion space (M, φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic *t*-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A)$$
.

To find a preimage (M, φ) of a space (M, α) over A consider M as an A[t]-module annihilated by t and replace $\alpha \colon M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. \Box

5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. Let A be an associative ring with involution, in which 2 is invertible. The homomorphism

$$\psi \colon W(A) \oplus W(A) \to W'(A[t, t^{-1}])$$

mapping (ξ, η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. There exists a homomorphism

Res:
$$W'(A[t, t^{-1}]) \rightarrow W(A)$$

with the following properties:

 R_1 : For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}]), Res(\xi) = 0.$

 R_2 : For any constant space $\xi \in W(A) \subset W'(A[t, t^{-1}])$, $Res(t \cdot \xi) = \xi$.

Proof. See Theorem 6.7.