

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE WITT GROUP OF LAURENT POLYNOMIALS  
**Autor:** Ojanguren, Manuel / Panin, Ivan  
**Kapitel:** 1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-64807>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## THE WITT GROUP OF LAURENT POLYNOMIALS

by Manuel OJANGUREN and Ivan PANIN

ABSTRACT. We give a direct, self-contained proof of the fact that for a large class of rings  $A$ , in particular for all regular rings with involution,  $W(A[t, 1/t]) = W(A) \oplus W(A)$ .

### 1. INTRODUCTION

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring  $A$ . These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on  $A$  and for their proofs we will use nothing more than a general result on the  $K$ -theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on  $A$  are necessary.

We begin by recalling briefly some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let  $A$  be an associative ring with an involution denoted by  $a \mapsto a^\circ$ . Except in §2 we will always assume that 2 is invertible in  $A$ . If  $M$  is a right  $A$ -module, we denote by  $M^*$  its dual  $\text{Hom}_A(M, A)$  endowed with the right action of  $A$  given by  $fa(x) = a^\circ f(x)$  for any  $f: M \rightarrow A$  and  $a \in A$ . If  $P$  is a finitely generated projective right  $A$ -module we identify it with  $P^{**}$  through the canonical isomorphism mapping  $x \in P$  to  $\hat{x}: P^* \rightarrow A$  defined by  $\hat{x}(f) = f(x)$ .

Let  $\epsilon$  be 1 or  $-1$ . An  $\epsilon$ -hermitian space over  $A$  is a pair  $(P, \alpha)$  consisting of a finitely generated projective right  $A$ -module  $P$  and an  $A$ -isomorphism  $\alpha: P \rightarrow P^*$  satisfying  $\alpha = \epsilon \alpha^*$ . For brevity  $\epsilon$ -hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring  $A$ ) is also called a *quadratic space*.

Two spaces  $(P, \alpha)$  and  $(Q, \beta)$  are *isometric* if there exists an  $A$ -isomorphism  $\varphi: P \rightarrow Q$  such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \alpha \downarrow & & \downarrow \beta \\ P^* & \xleftarrow{\varphi^*} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left( P \oplus P^*, \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \right).$$

The *orthogonal sum* of two spaces  $(P, \alpha)$  and  $(Q, \beta)$  is the space

$$(P, \alpha) \perp (Q, \beta) = (P \oplus Q, \alpha \oplus \beta).$$

If  $(P, \alpha)$  is a space and  $M$  a submodule of  $P$  we denote by  $M^\perp$  the orthogonal of  $M$ , defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^*,$$

where  $i^*$  is the dual of the inclusion  $i: M \rightarrow P$ . A submodule  $M$  of  $P$  is *totally isotropic* if  $M \subseteq M^\perp$ . A *sublagrangian* of a space  $(P, \alpha)$  is a totally isotropic direct factor of  $P$ . A *lagrangian* of  $(P, \alpha)$  is a sublagrangian  $L$  such that  $L = L^\perp$ . For instance,  $P$  and  $P^*$  are lagrangians of  $H(P)$ .

The Witt group  $W(A)$  of  $\epsilon$ -hermitian spaces over  $A$  is the quotient of the Grothendieck group of  $\epsilon$ -hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of  $W(A)$ .

Consider now the rings  $A[t]$  and  $A[t, t^{-1}]$ , endowed with the involution that fixes  $t$  and maps  $a \in A$  to  $a^\circ$ . For the ring  $A[t, t^{-1}]$  we introduce a variant  $W'(A[t, t^{-1}])$  of the Witt group. We first consider the Grothendieck group  $Q$  of  $\epsilon$ -hermitian spaces over  $A[t, t^{-1}]$  which are extended from  $A$  as  $A[t, t^{-1}]$ -modules, and its subgroup  $N$  generated by the hyperbolic spaces  $H(P)$  where  $P$  is extended from  $A$ . We then define  $W'(A[t, t^{-1}])$  as  $Q/N$ . Clearly  $W'(A[t, t^{-1}])$  maps canonically to  $W(A[t, t^{-1}])$ . Here are our results.

**A (THEOREM 3.1).** *Let  $A$  be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \rightarrow W(A[t])$$

*is an isomorphism.*

**B** (THEOREM 5.1). *Let  $A$  be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi: W(A) \oplus W(A) \rightarrow W'(A[t, t^{-1}])$$

*mapping  $(\xi, \eta)$  to  $\xi + t\eta$  is an isomorphism.*

**C** (THEOREM 7.1). *Let  $A$  be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi: W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$$

*be the canonical homomorphism.*

(a) *If  $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$ , then  $\varphi$  is surjective.*

(b) *If  $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$ , then  $\varphi$  is an isomorphism.*

Two examples will be constructed in §8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

**D** (PROPOSITION 6.8). *Let  $A$  be a commutative semilocal ring in which 2 is invertible. Let  $(P, \alpha)$  be a quadratic space over  $A$ . If  $(P, \alpha)$  is isometric to  $(P, t \cdot \alpha)$  over  $A[t, t^{-1}]$ , then  $(P, \alpha)$  is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$\text{Res}: W(A[t, t^{-1}]) \rightarrow W(A)$$

satisfying the following two properties:

- For any constant space  $\xi \in W(A) \subset W(A[t, t^{-1}])$ ,  $\text{Res}(\xi) = 0$ .
- For any constant space  $\xi \in W(A) \subset W(A[t, t^{-1}])$ ,  $\text{Res}(t \cdot \xi) = \xi$ .

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi \circ \psi: W(A) \oplus W(A) \rightarrow W(A[t, t^{-1}]),$$

which may fail, as in Example 8.1. However, there exists a residue map  $\text{Res}: W'(A[t, t^{-1}]) \rightarrow W(A)$  (Proposition 5.2) which yields the injectivity of  $\psi$ .

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let  $(P, \alpha)$  be any space. Then:*

1. *The space  $(P, \alpha) \perp (P, -\alpha)$  is hyperbolic.*
2. *If  $L$  is a lagrangian of  $(P, \alpha)$ , then  $(P, \alpha)$  is isometric to  $H(L)$ .*
3. *If  $M$  is a sublagrangian of  $(P, \alpha)$ , then the map  $\alpha$  induces on  $M^\perp/M$  a natural structure of hermitian space that makes it Witt equivalent to  $(P, \alpha)$ .*

## 2. $K$ -THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring  $A$  we denote by  $K_0(A)$  the Grothendieck group of finitely generated projective right  $A$ -modules and by  $K_1(A)$  the abelianized general linear group of  $A$ :  $K_1(A) = GL(A)/[GL(A), GL(A)]$ . By Whitehead's lemma  $K_1(A)$  is also the quotient of  $GL(A)$  by the subgroup  $E(A)$  generated by all elementary matrices over  $A$ .

For any functor  $F$  from rings to abelian groups we denote by  $N_+F(A)$  the kernel of the map  $F(A[t]) \rightarrow F(A)$  obtained by putting  $t = 0$ . Similarly, we denote by  $N_-F(A)$  the kernel of  $F(A[t^{-1}]) \rightarrow F(A)$  obtained by putting  $t^{-1} = 0$ . The inclusions of  $A[t]$  and  $A[t^{-1}]$  into  $A[t, t^{-1}]$  define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by  $LF(A)$ . The functor  $LK_1$  turns out to be naturally isomorphic to  $K_0$ , hence we will denote  $LK_i$  by  $K_{i-1}$  for  $i = 1$  and also for  $i = 0$ .

THEOREM 2.1. *Let  $A$  be any associative ring.*

(a) *For  $i = 0$  or  $1$  there exists a natural embedding*

$$\lambda_i: K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

*such that the composite*

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \rightarrow LK_i(A) = K_{i-1}(A)$$

*is the identity.*