

# 1. Introduction

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## THE WITT GROUP OF LAURENT POLYNOMIALS

by Manuel OJANGUREN and Ivan PANIN

ABSTRACT. We give a direct, self-contained proof of the fact that for a large class of rings  $A$ , in particular for all regular rings with involution,  $W(A[t, 1/t]) = W(A) \oplus W(A)$ .

### 1. INTRODUCTION

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring  $A$ . These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on  $A$  and for their proofs we will use nothing more than a general result on the  $K$ -theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on  $A$  are necessary.

We begin by recalling briefly some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let  $A$  be an associative ring with an involution denoted by  $a \mapsto a^\circ$ . Except in §2 we will always assume that 2 is invertible in  $A$ . If  $M$  is a right  $A$ -module, we denote by  $M^*$  its dual  $\text{Hom}_A(M, A)$  endowed with the right action of  $A$  given by  $fa(x) = a^\circ f(x)$  for any  $f: M \rightarrow A$  and  $a \in A$ . If  $P$  is a finitely generated projective right  $A$ -module we identify it with  $P^{**}$  through the canonical isomorphism mapping  $x \in P$  to  $\hat{x}: P^* \rightarrow A$  defined by  $\hat{x}(f) = f(x)$ .

Let  $\epsilon$  be 1 or  $-1$ . An  $\epsilon$ -hermitian space over  $A$  is a pair  $(P, \alpha)$  consisting of a finitely generated projective right  $A$ -module  $P$  and an  $A$ -isomorphism  $\alpha: P \rightarrow P^*$  satisfying  $\alpha = \epsilon\alpha^*$ . For brevity  $\epsilon$ -hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring  $A$ ) is also called a *quadratic space*.

Two spaces  $(P, \alpha)$  and  $(Q, \beta)$  are *isometric* if there exists an  $A$ -isomorphism  $\varphi: P \rightarrow Q$  such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \alpha \downarrow & & \downarrow \beta \\ P^* & \xleftarrow{\varphi^*} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left( P \oplus P^*, \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \right).$$

The *orthogonal sum* of two spaces  $(P, \alpha)$  and  $(Q, \beta)$  is the space

$$(P, \alpha) \perp (Q, \beta) = (P \oplus Q, \alpha \oplus \beta).$$

If  $(P, \alpha)$  is a space and  $M$  a submodule of  $P$  we denote by  $M^\perp$  the orthogonal of  $M$ , defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^*,$$

where  $i^*$  is the dual of the inclusion  $i: M \rightarrow P$ . A submodule  $M$  of  $P$  is *totally isotropic* if  $M \subseteq M^\perp$ . A *sublagrangian* of a space  $(P, \alpha)$  is a totally isotropic direct factor of  $P$ . A *lagrangian* of  $(P, \alpha)$  is a sublagrangian  $L$  such that  $L = L^\perp$ . For instance,  $P$  and  $P^*$  are lagrangians of  $H(P)$ .

The Witt group  $W(A)$  of  $\epsilon$ -hermitian spaces over  $A$  is the quotient of the Grothendieck group of  $\epsilon$ -hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of  $W(A)$ .

Consider now the rings  $A[t]$  and  $A[t, t^{-1}]$ , endowed with the involution that fixes  $t$  and maps  $a \in A$  to  $a^\circ$ . For the ring  $A[t, t^{-1}]$  we introduce a variant  $W'(A[t, t^{-1}])$  of the Witt group. We first consider the Grothendieck group  $Q$  of  $\epsilon$ -hermitian spaces over  $A[t, t^{-1}]$  which are extended from  $A$  as  $A[t, t^{-1}]$ -modules, and its subgroup  $N$  generated by the hyperbolic spaces  $H(P)$  where  $P$  is extended from  $A$ . We then define  $W'(A[t, t^{-1}])$  as  $Q/N$ . Clearly  $W'(A[t, t^{-1}])$  maps canonically to  $W(A[t, t^{-1}])$ . Here are our results.

**A (THEOREM 3.1).** *Let  $A$  be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \rightarrow W(A[t])$$

*is an isomorphism.*

**B** (THEOREM 5.1). *Let  $A$  be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi: W(A) \oplus W(A) \rightarrow W'(A[t, t^{-1}])$$

*mapping  $(\xi, \eta)$  to  $\xi + t\eta$  is an isomorphism.*

**C** (THEOREM 7.1). *Let  $A$  be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi: W'(A[t, t^{-1}]) \rightarrow W(A[t, t^{-1}])$$

*be the canonical homomorphism.*

(a) *If  $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$ , then  $\varphi$  is surjective.*

(b) *If  $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$ , then  $\varphi$  is an isomorphism.*

Two examples will be constructed in §8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

**D** (PROPOSITION 6.8). *Let  $A$  be a commutative semilocal ring in which 2 is invertible. Let  $(P, \alpha)$  be a quadratic space over  $A$ . If  $(P, \alpha)$  is isometric to  $(P, t \cdot \alpha)$  over  $A[t, t^{-1}]$ , then  $(P, \alpha)$  is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$\text{Res}: W(A[t, t^{-1}]) \rightarrow W(A)$$

satisfying the following two properties:

- For any constant space  $\xi \in W(A) \subset W(A[t, t^{-1}])$ ,  $\text{Res}(\xi) = 0$ .
- For any constant space  $\xi \in W(A) \subset W(A[t, t^{-1}])$ ,  $\text{Res}(t \cdot \xi) = \xi$ .

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi \circ \psi: W(A) \oplus W(A) \rightarrow W(A[t, t^{-1}]),$$

which may fail, as in Example 8.1. However, there exists a residue map  $\text{Res}: W'(A[t, t^{-1}]) \rightarrow W(A)$  (Proposition 5.2) which yields the injectivity of  $\psi$ .

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let  $(P, \alpha)$  be any space. Then:*

1. *The space  $(P, \alpha) \perp (P, -\alpha)$  is hyperbolic.*
2. *If  $L$  is a lagrangian of  $(P, \alpha)$ , then  $(P, \alpha)$  is isometric to  $H(L)$ .*
3. *If  $M$  is a sublagrangian of  $(P, \alpha)$ , then the map  $\alpha$  induces on  $M^\perp/M$  a natural structure of hermitian space that makes it Witt equivalent to  $(P, \alpha)$ .*

## 2. $K$ -THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring  $A$  we denote by  $K_0(A)$  the Grothendieck group of finitely generated projective right  $A$ -modules and by  $K_1(A)$  the abelianized general linear group of  $A$ :  $K_1(A) = GL(A)/[GL(A), GL(A)]$ . By Whitehead's lemma  $K_1(A)$  is also the quotient of  $GL(A)$  by the subgroup  $E(A)$  generated by all elementary matrices over  $A$ .

For any functor  $F$  from rings to abelian groups we denote by  $N_+F(A)$  the kernel of the map  $F(A[t]) \rightarrow F(A)$  obtained by putting  $t = 0$ . Similarly, we denote by  $N_-F(A)$  the kernel of  $F(A[t^{-1}]) \rightarrow F(A)$  obtained by putting  $t^{-1} = 0$ . The inclusions of  $A[t]$  and  $A[t^{-1}]$  into  $A[t, t^{-1}]$  define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by  $LF(A)$ . The functor  $LK_1$  turns out to be naturally isomorphic to  $K_0$ , hence we will denote  $LK_i$  by  $K_{i-1}$  for  $i = 1$  and also for  $i = 0$ .

THEOREM 2.1. *Let  $A$  be any associative ring.*

(a) *For  $i = 0$  or  $1$  there exists a natural embedding*

$$\lambda_i: K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

*such that the composite*

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \rightarrow LK_i(A) = K_{i-1}(A)$$

*is the identity.*