NOTE OF THE EDITORS

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concerning
"The Spectral Mapping Theorem, norms on rings, and resultants"
by D. LAKSOV, L. SVENSSON and A. THORUP

The Editors of *L'Enseignement Mathématique* have decided to publish here the referee's report, as it throws some interesting light on the authors' paper. At the referee's request, it appears without any editing as an anonymous contribution.

The main result, and many other related results, can be deduced from the fact that if one takes a generic square matrix of size n of indeterminates over the integers (call A_n the ring generated by the entries) and forces the characteristic polynomial to factor "generically", i.e., one adjoins n new indeterminates and equates the elementary symmetric functions of these with the coefficients of the characteristic polynomial (with suitable signs), then the resulting ring R_n is an integral domain. This takes a little proving and I do not know a formal reference, but it has been known to me for many years, and I strongly suspect that it is in the literature.

If one knows this (and it does not follow from trivial generalities about integral extensions — the authors are quite right about that), then, because R_n maps to every commutative ring such that a size n matrix has a split characteristic polynomial, all questions about polynomial identities in the entries of the matrix and its eigenvalues immediately reduce to the question of whether the identities hold in the generic case, i.e. for R_n. But since the generic ring is an integral domain, one may pass to its fraction field, which has characteristic 0, and even to the complex numbers. (The argument works even if the generic ring is merely reduced, i.e., has no nonzero nilpotents, since then it injects into a product of fields.)

Here is a sketch of a proof that R_n is a domain. First off, if one starts with any ring A and elements $a_1, \dots, a_n \in A$, and forms the quotient $A[Y_1, \dots, Y_n]/(e_i - a_i)$, where the e_i are the elementary symmetric functions

$Y_1 + \cdots + Y_n, \ldots, Y_1 \cdot \ldots \cdot Y_n$ of the Y_i, the quotient ring is a finitely generated free module over A, of rank $n!$. Killing $e_1 - a_1$ gives a monic equation of degree 1 for Y_1 in terms of the other variables. Then killing $e_2 - a_2$ gives a monic quadratic equation for Y_2 in terms of the remaining variables. Etc.

From this it follows that R_n is free of rank $n!$ over A_n. Thus, the issue of whether it is a domain is unaffected by inverting the nonzero elements of A_n, and one may pass to (i.e., tensor with) the fraction field B of A_n. Let L be the splitting field of the characteristic polynomial over B. It will suffice to see that the B-algebra homomorphism from $B \otimes R_n \to L$ that sends the images of the Y_i to the eigenvalues of the matrix in L is an isomorphism, and it is clearly onto.

Thus, it will be an isomorphism provided that L has degree $n!$ over B. We have reduced to showing that the splitting field of the characteristic polynomial of a generic matrix has degree $n!$ over the field generated by the entries. But this follows because the indeterminates can be specialized so as to give the companion matrix of any polynomial. (E.g., specialize the last column to elementary symmetric functions of n variables, the diagonal just below the main diagonal to all 1's, and the other entries to 0.)

So the short argument above (which would be shorter in a research paper version) shows that *every* polynomial identity with integer coefficients relating eigenvalues of a matrix and entries of that matrix which holds for matrices over the complex numbers holds in every commutative ring. There is one extra step needed for the Spectral Mapping Theorem: the coefficients of the polynomial F also need to be introduced as indeterminates, or instead one can point out that the identity really comes down to a finite number of identities saying that coefficients of polynomials in the extra variables are equal.

A simpler example is the Cayley-Hamilton theorem. It suffices to prove the result for a matrix of indeterminates over the integers, and so one is working over a polynomial ring over the integers. Here, one does not need to prove that any auxiliary ring is a domain. Since such a matrix is diagonalizable over an algebraically closed field containing the fraction field of the polynomial ring, one reduces the result for commutative rings to the case of diagonal matrices over a field, which is clear.