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7.2. COROLLARY (The Generalized Spectral Mapping Theorem). *Let N be a norm of degree n on a k -algebra A , and let α be an element of A . For all polynomials F in $k[x]$ we have the equations*

$$(7.2.1) \quad N_k(F(\alpha)) = \text{Res}(F, P_\alpha^N) = \prod_{i=1}^n F(\lambda_i)$$

in k , where $\lambda_1, \dots, \lambda_n$ are elements of any extension $R \supseteq k$ such that $P_\alpha^N(t) = \prod_{i=1}^n (t - \lambda_i)$ in $R[x]$.

In particular we have that $\text{Tr}^N(F(\alpha)) = \sum_{i=1}^n F(\lambda_i)$.

Proof. Let $P = P_\alpha^N$ be the characteristic polynomial of α with respect to N . The norm N restricts, via the canonical k -algebra homomorphism $k[x] \rightarrow A$ which sends x to α , to a norm on $k[x]$, and the characteristic polynomial of x with respect to this norm is P . On $k[x]$ we have the norm N , and the norms N'_p and N''_p of the Examples 6.1 and 6.2, and the characteristic polynomial of x with respect to all three norms is P . It follows from the Theorem that these three norms are equal. The equations (7.2.1) express the equality of the norms applied to the polynomial $F(x)$. Finally the expression for the trace follows by considering the coefficient of t^{n-1} of the left and right side of (7.2.1) applied to the polynomial $t - F(x)$ in $k[t][x]$. \square

The formula $\text{Res}(F, P) = \prod_{i=1}^n F(\lambda_i)$ of Corollary (7.2) is the generalization to rings of the well-known interpretation of resultants by the roots of the monic polynomial P in the case when k is a field. If F is also monic and $F = \prod_{j=1}^m (x - \mu_j)$ in $R[x]$ we have

$$\text{Res}(F, P) = \prod_{i=1}^n F(\lambda_i) = \prod_{i=1}^n \prod_{j=1}^m (\lambda_i - \mu_j)$$

which is often used as a definition of the resultant in the case k is an algebraically closed field.

8. THE DISCRIMINANT

We shall use the Generalized Spectral Mapping Theorem of Section 7 to prove two results on discriminants that are well-known for algebras of finite dimension over fields (see e.g. [B2], §5, Corollaire 6 and Corollaire 7, p.38). Note that k below, as above, denotes a commutative ring with unity.

Let P be a monic polynomial in $k[x]$ of degree n . The *discriminant* of P is the element $(-1)^{n(n-1)/2} \text{Res}(P', P)$ of k , where P' is the derivative of P .

Let $k \subseteq k' = k[\lambda_1, \dots, \lambda_n]$ be the canonical extension constructed in Example 6.2. We write $\Delta(P) = \prod_{i>j} (\lambda_i - \lambda_j)$. The formula for the resultant of Corollary 7.2 gives the equations

$$(-1)^{n(n-1)/2} \text{Res}(P', P) = (-1)^{n(n-1)/2} \prod_{i \neq j} (\lambda_i - \lambda_j) = \prod_{i>j} (\lambda_i - \lambda_j)^2 = \Delta(P)^2,$$

hence $\Delta(P)^2$ is the discriminant of P .

8.1. PROPOSITION. *Let N be a norm of degree n on a k -algebra A , and let α be an element of A . Denote by P_α the characteristic polynomial of α with respect to N . Then we have the equations in k :*

$$\det \text{Tr}^N(\alpha^{p+q})_{p,q=0,\dots,n-1} = \Delta(P_\alpha)^2 = (-1)^{n(n-1)/2} N_k(P'_\alpha(\alpha)).$$

Proof. By (7.2.1) we have the equation $N_k(P_\alpha(\alpha)) = \text{Res}(P'_\alpha, P_\alpha)$. Hence the second equation holds.

The rest of the proof is classical, as given in [B2]. We note that $\prod_{i>j} (\lambda_i - \lambda_j)$ is the determinant of the matrix (λ_i^q) with row number $i = 1, \dots, n$ and column number $q = 0, \dots, n - 1$. When this matrix is multiplied from the left by its transpose the entry in position p, q is equal to the sum $\sum_{i=1}^n \lambda_i^{p+q}$, and consequently equal to $\text{Tr}^N(\alpha^{p+q})$. Hence we have proved the Proposition. \square

8.2. PROPOSITION. *Let N be a norm on a k -algebra A , and let α be an element in A . In the ring of power series in the variable t with coefficients in k we have the equation:*

$$\left(\frac{d}{dt} \log\right) N_{k[t]}(1 - t\alpha) := \frac{d}{dt} N_{k[t]}(1 - t\alpha) / N_{k[t]}(1 - t\alpha) = - \sum_{j=0}^{\infty} \text{Tr}^N(\alpha^{j+1}) t^j.$$

Proof. Let n be the degree of the norm N . We have

$$N_{k[t]}(1 - t\alpha) = \prod_{i=1}^n (1 - \lambda_i t)$$

and consequently we have equations

$$(d \log / dt) N_{k[t]}(1 - t\alpha) = \sum_{i=1}^n -\lambda_i / (1 - t\lambda_i) = - \sum_{j=0}^{\infty} \sum_{i=1}^n \lambda_i^{j+1} t^j.$$

It follows from Corollary 7.2 that $\text{Tr}^N(\alpha^l) = \sum_{i=1}^n \lambda_i^l$ for $l = 1, 2, \dots$. The formula of the Proposition follows. \square

We note that, when k contains the rational numbers and N is a norm of degree n on a k -algebra A , we have that the characteristic polynomial $N_{k[t]}(t - \alpha)$ of an element α of A is determined by the elements $\text{Tr}^N(\alpha^j)$ for $j = 1, \dots, n$.

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