§4. HISTORICAL REMARKS

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REMARK. We observe that conversely, Theorem 1 can be deduced from Theorem 2; we shall not elaborate on this; however, our proof of Theorem 1 shows that its validity stems from a simple general result on L^p-spaces.

§4. HISTORICAL REMARKS

The inequality (1) was given first by Hausdorff [H] in 1923 for the groups $G = \mathbf{T}$ (with $\widehat{G} = \mathbf{Z}$) and $G = \mathbf{Z}$ (with $\widehat{G} = \mathbf{T}$). Hausdorff was inspired by the work of W. H. Young from 1912–13 who proved that the Fourier series of a function in L^p, $1 \leq p \leq 2$, had coefficients which were in $\ell^{p'}$ (and, in a suitable sense, vice-versa) for $p' = 2k$, a positive even integer, $p = 2k/(2k-1)$. Young did not formulate his results in terms of inequalities which were given first by Hausdorff (for all $p \in [1,2]$ and for the groups $G = \mathbf{T}$, $G = \mathbf{Z}$, i.e. for Fourier series). Hausdorff's proof, which is all but forgotten today, used Young's results for $p' = 2k$ and some of Young's techniques to carry out an interpolation argument for all the values of p, p', $1 \leq p \leq 2$, missing in Young's work. Hausdorff's paper [H] gives the exact references to W. H. Young's paper which were related to his work.

Shortly afterwards, after having heard of Hausdorff's inequalities, F. Riesz obtained independently (in [RF]) some Hausdorff-Young type inequalities, valid for series expansions in terms of arbitrary *bounded* orthogonal functions. This paper of F. Riesz was important not only because it showed that Hausdorff-Young type inequalities did not belong exclusively to the theory of Fourier series but also because F. Riesz (in collaboration with his colleague A. Haar) conjectured there the validity of a general "arithmetical" inequality for linear forms (in a finite number of variables) which they claimed to be enough for proving F. Riesz's theorem for orthogonal expansions.

It was this conjecture which seems to have led M. Riesz (F. Riesz's younger brother) to formulate and prove in 1927 ([RM]) his convexity theorem for bilinear forms and use it to deduce Hausdorff-Young-F. Riesz inequalities and many others. M. Riesz's work was exactly what A. Weil used in 1940 to establish (1) for general locally compact commutative groups in his book [W], p. 117. As is well-known, once the Plancherel theorem for a general $L^2(G)$, G locally compact commutative, is established (and this was done by Weil) the proof of (1) via M. Riesz's theorem is almost immediate. M. Riesz's work was simplified and much generalized by Thorin in 1938 (and later in 1948; exact references can be found in [DS] or in [HR]) which launched the later theory of interpolation of operators due to many well-known mathematicians which we shall not attempt to describe here. As regards Theorem 1 given in §3, it was proven by I. Segal in 1950 (for the part concerning $C_0(\widehat{G})$) and generally by E. Hewitt in 1954. The theorem was rediscovered by Rajagopalan in 1964; exact references to the papers of these authors can be found in [HR] vol. 2. The fact that the inequality (1) does not generalize to $p > 2$ had been foreseen in papers of 1918–19 by Carleman, Hardy and Littlewood, Landau for the case of $G = \mathbf{T}$ (exact references are in Hausdorff's paper [H]) where the work depends on the detailed study of the Fourier series of special continuous functions. However, I do not know of any explicit previous formulation and proof of Theorem 2 for arbitrary infinite locally compact commutative groups G; it is difficult to imagine that it has not been written down somewhere, since its proof is a straight-forward deduction from the inversion formula and the non-surjectivity theorem.

Theorem 1 has been generalized to the case of non-commutative compact groups G in [HR] vol. 2, (37.19), p. 429; now, $\widehat{G}$ is taken to be the set of all equivalence classes of continuous unitary irreducible representations of G. For G any locally compact unimodular group, Kunze (1958) has given the appropriate formulation of the Hausdorff-Young inequality (1) (see reference in [HR] vol. 2). If G is compact, $\widehat{G}$ as a set has the discrete topology and our proof of Theorem 1 carries over to this case. Theorem 2 in this case can be formulated as in [HR] vol. 2, (37.19) (iii), p. 429; its proof now is no more difficult than that of our Thm. 2. However, we do not intend to discuss the non-commutative case in any detail here.

We close this section by mentioning the remarkable later (1990) development around the Hausdorff-Young inequality due to Lieb [L]. Lieb has shown, generalizing considerably previous work of Babenko (1961) and Beckner (1975), that for the group $\mathbf{R}^n$ and for "Gaussian" transforms T more general than the Fourier transform, one has

$$\|Tf\|_{p'} \leq M_p \|f\|_p \tag{6}$$

where $M_p < 1$; Lieb has determined M_p exactly and has specified all the functions f for which equality obtains in (11). In particular, the L^p-Fourier transform in $\mathbf{R}^n$, $(1 \leq p < 2)$ turns out to be a strict contraction (a fact noticed by Babenko and Beckner) whose contraction coefficient (< 1) can be determined exactly; this is in sharp contrast to the situation in those locally compact abelian groups which have compact open subgroups where the corresponding Fourier transforms are just contractions. This has been studied in detail by Hewitt, Hirschmann, Ross ([HR] vol. 2, §43).