Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
Autor: BAUM, Paul / CONNES, Alain
Kapitel: 10. FURTHER DEVELOPMENTS
DOI: https://doi.org/10.5169/seals-64793

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 23.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

DEFINITION 6. $K^*(V,F) = \Gamma(V,F)/\sim$. Addition in $K^*(V,F)$ is by disjoint union of K-cocycles. The natural homomorphism of abelian groups

$$K^i(V,F) \to K_i\, C^*(V,F)$$

is defined by

$$(Z,\xi) \to \mu(Z,\xi)\,.$$

CONJECTURE. *$\mu\colon K^*(V,F) \to K_* C^*(V,F)$ is an isomorphism.*

REMARK 7. Calculations of M. Pennington [25] and A.M. Torpe [32] verify the conjecture for certain foliations.

Given (V,F), let BG be the classifying space of the holonomy groupoid G. Since ν is a G-vector bundle on V, ν induces a vector bundle τ on BG. As in §3 above there is then a natural map

$$K_*^\tau(BG) \to K^*(V,F)\,.$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \to K^*(V,F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \to K^*(V,F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \to K^*(V,F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu\colon K^*(V,F) \to K_* C^*(V,F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^*-algebra. If G is a Lie group and X is a G-manifold, then using A as coefficients there is both a geometric and an analytic K-theory for (X,G). The analytic K-theory is the K-theory of the C^*-algebra $(C_0(X)\rtimes G)\otimes A$.

The geometric K-theory is obtained from K-cocycles (Z, ξ, f) where Z, f are as in §2 and $\xi = \{E_0 \xrightarrow{\sigma} E_1\}$ uses G-vector bundles E_0, E_1 on $T^*Z \oplus f^*T^*X$ such that the fibres of E_i are finitely generated projective modules over A. Denote this geometric K-theory by $K^*(X, G; A)$. The natural map

$$K^i(X, G; A) \to K_i[(C_0(X) \rtimes G) \otimes A]$$

is defined by using elliptic operators in the spirit of Miscenko-Fomenko [22]. We conjecture that this natural map is an isomorphism.

In the notation of Kasparov [18] the group denoted here by $K_*[C_0(X) \rtimes G]$ is $KK(\mathbf{C}, C_0(X) \rtimes G)$. For the K-homology group $KK(C_0(X) \rtimes G, \mathbf{C})$ there is a geometric group $K_*(X, G)$ which is the G-equivariant version of the topologically defined K-homology of [9]. Using transversally elliptic operators [2] one then obtains a natural map

$$K_*(X, G) \to KK(C_0(X) \rtimes G, \mathbf{C}).$$

We conjecture that this map is injective and that its image is dense (with respect to the natural topology) in $KK(C_0(X) \rtimes G, \mathbf{C})$.

11. Acknowledgements

We are grateful to A. Borel for communicating us

PROPOSITION 1 (A. Borel [10]). *Let G be a Lie group with $\pi_0 G$ finite and maximal compact subgroup H. If Z is any proper G-manifold then there exists a G-map from Z to $H\backslash G$.*

In §5 above this was proved for G a connected semi-simple Lie group with finite center. By the argument of §5, Borel's result implies:

COROLLARY 2. *Let G be a Lie group with $\pi_0 G$ finite. For any G-manifold X there is an isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h}\backslash\mathfrak{g})^*) \to K^i(X, G) \qquad (i = 0, 1).$$

Finally, it is a pleasure to thank the many mathematicians who have given us stimulating and enlightening comments. We thank A. Borel, J. Cohen, R. Douglas, B. Lawson, G. Mackey, R. MacPherson, J. Rosenberg, and G. Schwarz.