

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
Autor: BAUM, Paul / CONNES, Alain
Kapitel: 9. Foliations
DOI: <https://doi.org/10.5169/seals-64793>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

COROLLARY 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of \mathbf{R} , $\Delta = \text{tr}_\Gamma(K_0(C_r^*(\Gamma, \gamma)))$ contains the group:*

$$\langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle .$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma, \mathbf{R})/H^2(\Gamma, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C_r^*(\Gamma, \gamma))$ by the canonical trace $\zeta = \text{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma, \gamma)$ (for any value of γ), one gets that the corresponding C^* -algebras are isomorphic only when the Γ 's are the same (using K_1) and when the γ 's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞ -manifold, and let F be a C^∞ -foliation of V . Thus F is a C^∞ -integrable sub-vector bundle of TV . As in [33] let G be the holonomy groupoid (graph) of (V, F) . The manifold V is assumed to be Hausdorff and second countable. G , however, is a C^∞ -manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞ -paths

$$\gamma: [0, 1] \rightarrow V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0, 1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{r} \end{smallmatrix} V$.

Let Z be a possibly non-Hausdorff C^∞ -manifold. Assume given a C^∞ -map $\rho: Z \rightarrow V$, set

$$Z \circ G = \{(z, \gamma) \in Z \times G \mid \rho(z) = s(\gamma)\} .$$

A C^∞ right action of G on Z is a C^∞ -map

$$Z \circ G \rightarrow Z$$

denoted by

$$(z, \gamma) \rightarrow z\gamma$$

such that

$$\rho(z\gamma) = r(\gamma), \quad (z\gamma)\gamma' = z(\gamma\gamma'), \quad (zl_p) = z,$$

where l_p denotes the constant path at $p \in V$.

An action of G on Z is *proper* if:

- (i) the map $Z \circ G \rightarrow Z \times Z$ given by $(z, \gamma) \mapsto (z, z\gamma)$ is proper (i.e. the inverse image of a compact set is compact);
- (ii) the quotient space Z/Γ is Hausdorff. Here Z/Γ is the set of equivalence classes of $z \in Z$ where $z \sim z'$ if, for some $\gamma \in G$, $z\gamma = z'$.

Specializing to $Z = V$, the groupoid G acts on V by $\rho(p) = p$ and

$$p\gamma = \gamma(1)$$

($p \in V$, $\gamma \in G$, $p = \gamma(0)$). For many examples this action of G on V is not proper. Set $\nu_p = T_p V / F_p$, so that ν is the normal bundle of the foliation. ν is a G -vector bundle since the derivative of holonomy gives a linear map

$$\nu_p \mapsto \nu_{p\gamma}.$$

This is, of course, just the well-known fact that ν is flat along the leaves of the foliation.

More generally, if Z is a G -manifold, then the orbits of the G -action foliate Z . Denote the normal bundle of this foliation by $\tilde{\nu}$. Then $\tilde{\nu}$ is a G -vector bundle on Z .

If Z is a proper G -manifold, a G -vector bundle on Z with G -compact support is a triple (E_0, E_1, σ) where E_0, E_1 are G -vector bundles on Z and $\sigma: E_0 \rightarrow E_1$ is a morphism of G -vector bundles with $\text{Support}(\sigma)$ G -compact. As in §2 above one then defines $V_G^i(Z)$ and $K_G^i(Z)$, $i = 0, 1$. These are defined and used *only* for proper G -manifolds.

DEFINITION 1. A K -cocycle for (V, F) is a pair (Z, ξ) such that

- (1) Z is a proper G -manifold,
- (2) $\xi \in V_G^*[(\tilde{\nu})^* \oplus \rho^* \nu^*]$, where $\rho: Z \rightarrow V$ is given by the action of G on Z .

In [12] and [14] a canonical C^* -algebra $C^*(V, F)$ is constructed. This C^* -algebra can heuristically be thought of (up to Morita equivalence) as the

algebra of continuous functions on the "space of leaves" of the foliation. Thus $K_*C^*(V, F)$ can be viewed as the K -theory of the "space of leaves" of the foliation.

To define the geometric K -theory $K^*(V, F)$ we proceed quite analogously to §2 above.

THEOREM 2. *Let (Z, ξ) be a cocycle for (V, F) . Then (Z, ξ) determines an element in $K_*C^*(V, F)$.*

Proof. If $\rho: Z \rightarrow V$ is a submersion then ξ gives rise to the symbol of a G -equivariant family of elliptic operators D , parametrized by the points of V . The K -theory index of this family D is the desired element of $K_*C^*(V, F)$.

If $\rho: Z \rightarrow V$ is not a submersion, then as in the proof of Theorem 1 of §2 one reduces to the submersion case. \square

REMARK 3. With D as in the proof of the Theorem, $\text{Index}(D) \in K_*C^*(V, F)$ will be denoted $\mu(Z, \xi)$. For $\xi \in V_G^i[(\tilde{\nu})^* \oplus \rho^* \nu^*]$, $\mu(Z, \xi) \in K_i C^*(V, F)$, $i = 0, 1$.

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ \rho_1 \searrow & & \swarrow \rho_2 \\ & V & \end{array}$$

where Z_1, Z_2 are G -manifolds with Z_1, Z_2 proper and h is a G -map. There is then a Gysin map

$$h_!: K_G^i[(\tilde{\nu}_1)^* \oplus \rho_1^* \nu^*] \rightarrow K_G^i[(\tilde{\nu}_2)^* \oplus \rho_2^* \nu^*].$$

THEOREM 4. *If $\xi_1 \in V_G^*[(\tilde{\nu}_1)^* \oplus \rho_1^* \nu]$ then $\mu(Z_1, \xi_1) = \mu(Z_2, h_!(\xi_1))$.*

REMARK 5. Let $\Gamma(V, F)$ be the collection of all K -cocycles (Z, ξ) for (V, F) . On $\Gamma(V, F)$ impose the equivalence relation \sim , where $(Z, \xi) \sim (Z', \xi')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z'' & \xleftarrow{h'} & Z' \\ \rho \searrow & & \downarrow \rho'' & & \swarrow \rho' \\ & & V & & \end{array}$$

with h and h' G -maps and with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 6. $K^*(V, F) = \Gamma(V, F)/\sim$. Addition in $K^*(V, F)$ is by disjoint union of K -cocycles. The natural homomorphism of abelian groups

$$K^i(V, F) \rightarrow K_i C^*(V, F)$$

is defined by

$$(Z, \xi) \rightarrow \mu(Z, \xi).$$

CONJECTURE. $\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$ is an isomorphism.

REMARK 7. Calculations of M. Pennington [25] and A. M. Torpe [32] verify the conjecture for certain foliations.

Given (V, F) , let BG be the classifying space of the holonomy groupoid G . Since ν is a G -vector bundle on V , ν induces a vector bundle τ on BG . As in §3 above there is then a natural map

$$K_*^\tau(BG) \rightarrow K^*(V, F).$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \rightarrow K^*(V, F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \rightarrow K^*(V, F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \rightarrow K^*(V, F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^* -algebra. If G is a Lie group and X is a G -manifold, then using A as coefficients there is both a geometric and an analytic K -theory for (X, G) . The analytic K -theory is the K -theory of the C^* -algebra $(C_0(X) \rtimes G) \otimes A$.