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COROLLARY 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of \mathbf{R} , $\Delta = \text{tr}_\Gamma(K_0(C_r^*(\Gamma, \gamma)))$ contains the group:*

$$\langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle .$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma, \mathbf{R})/H^2(\Gamma, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C_r^*(\Gamma, \gamma))$ by the canonical trace $\zeta = \text{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma, \gamma)$ (for any value of γ), one gets that the corresponding C^* -algebras are isomorphic only when the Γ 's are the same (using K_1) and when the γ 's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞ -manifold, and let F be a C^∞ -foliation of V . Thus F is a C^∞ -integrable sub-vector bundle of TV . As in [33] let G be the holonomy groupoid (graph) of (V, F) . The manifold V is assumed to be Hausdorff and second countable. G , however, is a C^∞ -manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞ -paths

$$\gamma: [0, 1] \rightarrow V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0, 1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \begin{matrix} \xrightarrow{s} \\ \xrightarrow{r} \end{matrix} V$.

Let Z be a possibly non-Hausdorff C^∞ -manifold. Assume given a C^∞ -map $\rho: Z \rightarrow V$, set

$$Z \circ G = \{(z, \gamma) \in Z \times G \mid \rho(z) = s(\gamma)\} .$$

A C^∞ right action of G on Z is a C^∞ -map

$$Z \circ G \rightarrow Z$$

denoted by

$$(z, \gamma) \rightarrow z\gamma$$

such that

$$\rho(z\gamma) = r(\gamma), \quad (z\gamma)\gamma' = z(\gamma\gamma'), \quad (zl_p) = z,$$

where l_p denotes the constant path at $p \in V$.

An action of G on Z is *proper* if:

- (i) the map $Z \circ G \rightarrow Z \times Z$ given by $(z, \gamma) \mapsto (z, z\gamma)$ is proper (i.e. the inverse image of a compact set is compact);
- (ii) the quotient space Z/Γ is Hausdorff. Here Z/Γ is the set of equivalence classes of $z \in Z$ where $z \sim z'$ if, for some $\gamma \in G$, $z\gamma = z'$.

Specializing to $Z = V$, the groupoid G acts on V by $\rho(p) = p$ and

$$p\gamma = \gamma(1)$$

($p \in V$, $\gamma \in G$, $p = \gamma(0)$). For many examples this action of G on V is not proper. Set $\nu_p = T_p V / F_p$, so that ν is the normal bundle of the foliation. ν is a G -vector bundle since the derivative of holonomy gives a linear map

$$\nu_p \mapsto \nu_{p\gamma}.$$

This is, of course, just the well-known fact that ν is flat along the leaves of the foliation.

More generally, if Z is a G -manifold, then the orbits of the G -action foliate Z . Denote the normal bundle of this foliation by $\tilde{\nu}$. Then $\tilde{\nu}$ is a G -vector bundle on Z .

If Z is a proper G -manifold, a G -vector bundle on Z with G -compact support is a triple (E_0, E_1, σ) where E_0, E_1 are G -vector bundles on Z and $\sigma: E_0 \rightarrow E_1$ is a morphism of G -vector bundles with $\text{Support}(\sigma)$ G -compact. As in §2 above one then defines $V_G^i(Z)$ and $K_G^i(Z)$, $i = 0, 1$. These are defined and used *only* for proper G -manifolds.

DEFINITION 1. A K -cocycle for (V, F) is a pair (Z, ξ) such that

- (1) Z is a proper G -manifold,
- (2) $\xi \in V_G^*[(\tilde{\nu})^* \oplus \rho^* \nu^*]$, where $\rho: Z \rightarrow V$ is given by the action of G on Z .

In [12] and [14] a canonical C^* -algebra $C^*(V, F)$ is constructed. This C^* -algebra can heuristically be thought of (up to Morita equivalence) as the

algebra of continuous functions on the “space of leaves” of the foliation. Thus $K_*C^*(V, F)$ can be viewed as the K -theory of the “space of leaves” of the foliation.

To define the geometric K -theory $K^*(V, F)$ we proceed quite analogously to §2 above.

THEOREM 2. *Let (Z, ξ) be a cocycle for (V, F) . Then (Z, ξ) determines an element in $K_*C^*(V, F)$.*

Proof. If $\rho: Z \rightarrow V$ is a submersion then ξ gives rise to the symbol of a G -equivariant family of elliptic operators D , parametrized by the points of V . The K -theory index of this family D is the desired element of $K_*C^*(V, F)$.

If $\rho: Z \rightarrow V$ is not a submersion, then as in the proof of Theorem 1 of §2 one reduces to the submersion case. \square

REMARK 3. With D as in the proof of the Theorem, $\text{Index}(D) \in K_*C^*(V, F)$ will be denoted $\mu(Z, \xi)$. For $\xi \in V_G^i [(\tilde{\nu})^* \oplus \rho^* \nu^*]$, $\mu(Z, \xi) \in K_i C^*(V, F)$, $i = 0, 1$.

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ \rho_1 \searrow & & \swarrow \rho_2 \\ & V & \end{array}$$

where Z_1, Z_2 are G -manifolds with Z_1, Z_2 proper and h is a G -map. There is then a Gysin map

$$h_!: K_G^i [(\tilde{\nu}_1)^* \oplus \rho_1^* \nu^*] \rightarrow K_G^i [(\tilde{\nu}_2)^* \oplus \rho_2^* \nu^*].$$

THEOREM 4. *If $\xi_1 \in V_G^* [(\tilde{\nu}_1)^* \oplus \rho_1^* \nu]$ then $\mu(Z_1, \xi_1) = \mu(Z_2, h_!(\xi_1))$.*

REMARK 5. Let $\Gamma(V, F)$ be the collection of all K -cocycles (Z, ξ) for (V, F) . On $\Gamma(V, F)$ impose the equivalence relation \sim , where $(Z, \xi) \sim (Z', \xi')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z'' & \xleftarrow{h'} & Z' \\ \rho \searrow & & \downarrow \rho'' & & \swarrow \rho' \\ & & V & & \end{array}$$

with h and h' G -maps and with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 6. $K^*(V, F) = \Gamma(V, F)/\sim$. Addition in $K^*(V, F)$ is by disjoint union of K -cocycles. The natural homomorphism of abelian groups

$$K^i(V, F) \rightarrow K_i C^*(V, F)$$

is defined by

$$(Z, \xi) \rightarrow \mu(Z, \xi).$$

CONJECTURE. $\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$ is an isomorphism.

REMARK 7. Calculations of M. Pennington [25] and A. M. Torpe [32] verify the conjecture for certain foliations.

Given (V, F) , let BG be the classifying space of the holonomy groupoid G . Since ν is a G -vector bundle on V , ν induces a vector bundle τ on BG . As in §3 above there is then a natural map

$$K_*^\tau(BG) \rightarrow K^*(V, F).$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \rightarrow K^*(V, F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \rightarrow K^*(V, F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \rightarrow K^*(V, F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^* -algebra. If G is a Lie group and X is a G -manifold, then using A as coefficients there is both a geometric and an analytic K -theory for (X, G) . The analytic K -theory is the K -theory of the C^* -algebra $(C_0(X) \rtimes G) \otimes A$.