

|                     |   |
|---------------------|---|
| <b>Zeitschrift:</b> | L'Enseignement Mathématique   |
| <b>Herausgeber:</b> | Commission Internationale de l'Enseignement Mathématique                              |
| <b>Band:</b>        | 46 (2000)   |
| <b>Heft:</b>        | 3-4: L'ENSEIGNEMENT MATHÉMATIQUE  |
| <br>                |   |
| <b>Artikel:</b>     | ORDERINGS OF MAPPING CLASS GROUPS AFTER THURSTON                                      |
| <b>Autor:</b>       | SHORT, Hamish / Wiest, Bert   |
| <b>Kapitel:</b>     | 3. Main results   |
| <b>DOI:</b>         | <a href="https://doi.org/10.5169/seals-64801">https://doi.org/10.5169/seals-64801</a> |

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 23.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

and  $1_{B_n} < \varphi$ . In this ordering the commutator subgroup is convex [19], and we leave it to the reader to verify that no Thurston-type ordering has this property.

### 3. MAIN RESULTS

We shall mainly be interested in the case  $S = D_n$  ( $n \geq 2$ ), where  $D_n$  is the closed unit disk in  $\mathbf{C}$ , with  $n$  punctures lined up in the real interval  $(-1, 1)$ ; in this case the mapping class group is a braid group:  $\mathcal{MCG}(D_n) = B_n$ . We recall that for  $\alpha \in (0, \pi)$  we denote by  $\gamma_\alpha$  the geodesic which starts at the basepoint with angle  $\alpha$  with  $\partial S$ , and by  $\tilde{\gamma}_\alpha$  its preimage in the universal cover starting at the basepoint of  $S^\sim$ .

**DEFINITION 3.1.** A geodesic  $\gamma_\alpha$ ,  $\alpha \in (0, \pi)$ , is said to be of *finite type* if it satisfies at least one of the following conditions:

- (a) there exists a finite initial segment  $\gamma_\alpha^t$  such that any two punctures that lie in the same path component of  $S \setminus \gamma_\alpha^t$  also lie in the same path component of  $S \setminus \gamma_\alpha$ , or
- (b) it falls into a puncture, or
- (c) it spirals towards a simple closed geodesic.

If a geodesic  $\gamma_\alpha$  is not of finite type then we say it is of *infinite type*. We also define the ordering of  $\mathcal{MCG}(S)$  induced by a geodesic  $\gamma_\alpha$  to be of finite or infinite type if  $\gamma_\alpha$  is of finite or infinite type.

An infinite type geodesic looks as follows. All its self intersections occur in some finite initial segment  $\gamma_\alpha^t$ . At least one of the path components of  $S \setminus \gamma_\alpha^t$  contains three or more punctures in its interior, and the closure of  $\gamma_\alpha \setminus \gamma_\alpha^t$  is a geodesic lamination without closed leaves inside such a component. In particular, there is a pair of punctures which are separated by the whole geodesic, but not by any finite initial segment. (Note that the geodesic  $\gamma_\alpha \setminus \gamma_\alpha^t$  is isolated from both sides – in this it is very different from leaves of geodesic laminations on surfaces without boundary.)

**DEFINITION 3.2.** For a geodesic  $\gamma_\alpha$  of finite respectively infinite type we say that it *fills the surface in finite* respectively *infinite time* if all punctures lie in different path components of  $S \setminus \gamma_\alpha$ . By contrast, a geodesic  $\gamma_\alpha$  *does not fill* the surface if  $S \setminus \gamma_\alpha$  has a path component that contains two punctures.

The aim of the rest of the paper is to prove the following theorems. Recall that every point  $\alpha \in (0, \pi)$  gives rise to a – possibly partial – ordering of  $\mathcal{MCG}(S)$ . The first theorem gives criteria for these orderings to be total or, equivalently, for the orbit of  $\alpha$  to be free.

**THEOREM 3.3.** *Let  $S$  be any hyperbolic surface.*

(a) *If a geodesic  $\gamma_\alpha$  does not fill  $S$ , then the orbit of  $\alpha \in (0, \pi)$  is not free.*

(b) *If  $\gamma_\alpha$  is of finite type, then the converse holds as well: if  $\gamma_\alpha$  fills the surface, then  $\alpha$  has free orbit.*

(c) *Let  $\mathcal{I} := \{\alpha \mid \gamma_\alpha \text{ is of infinite type}\} \subset (0, \pi)$ . Then  $\mathcal{I}$  is uncountable, and all but countably many of its elements have free orbits. In any neighbourhood of an  $\alpha \in \mathcal{I}$  there exist points of both finite and infinite type, i.e. there are  $\alpha' \neq \alpha$  and  $\beta \in (0, \pi)$  such that  $\gamma_{\alpha'} \in \mathcal{I}$  and  $\gamma_\beta \notin \mathcal{I}$ .*

The next theorem gives a classification of orders of Thurston-type.

**THEOREM 3.4.** *If  $S$  is a punctured disk, we have:*

(a) *An ordering cannot be both of finite and infinite type.*

(b) *Given two geodesics  $\gamma_\alpha, \gamma_\beta$  of finite type, one can decide whether or not they determine the same ordering.*

(c) *Given two geodesics  $\gamma_\alpha, \gamma_\beta$  of infinite type, one can decide whether or not they determine the same ordering. For instance, if  $\gamma_\alpha$  and  $\gamma_\beta$  are embedded, then they determine the same ordering if and only if  $\beta = \Delta^{2k}(\alpha)$  for some  $k \in \mathbf{Z}$  (i.e. if  $\gamma_\beta$  is obtained from  $\gamma_\alpha$  by sliding the starting point  $2k$  times around  $\partial D_n$ ).*

(Note that part (a) is not immediately clear: it is conceivable that finite and infinite type geodesics induce the same orderings.) In fact, we shall develop machinery which gives a very good and explicit understanding of finite type orderings:

**THEOREM 3.5.** *There are only finitely many conjugacy classes of orderings of finite type of  $\mathcal{MCG}(D_n) = B_n$ . The number  $N_n$  of conjugacy classes can be calculated by the following recursive formula*

$$N_2 = 1 \quad \text{and} \quad N_n = N_{n-1} + \sum_{k=2}^{n-2} \binom{n-2}{k-1} N_k N_{n-k}.$$

We do not know if there exists a “closed” formula for  $N_n$ . The following list gives the first few values:

| $n$   | 2 | 3 | 4 | 5 | 6  | 7   | 8    |
|-------|---|---|---|---|----|-----|------|
| $N_n$ | 1 | 1 | 3 | 9 | 39 | 189 | 1197 |

Theorems 3.4 and 3.5 almost certainly generalise to mapping class groups of other negatively curved surfaces, but in order to keep our machinery simple, we stick to the special case of punctured disks.

#### 4. ORDERINGS OF MAPPING CLASS GROUPS USING CURVE DIAGRAMS

In this section we present another method for constructing left orderings on  $B_n$ , using certain diagrams on  $D_n$ , which we call *curve diagrams*. Both the definition of curve diagrams and the orderings associated to them are generalisations of similar concepts in [9].

**CONVENTION.** Whenever we talk about geodesics in  $D_n$ , we think of the punctures as being holes in the disk, whose neighbourhoods on the disk have the geometry of cusps. By contrast, when we talk about curve diagrams, we think of the punctures as distinguished points on, and belonging to, the disk, and we ignore the geometric structure. This changing perspective should not cause confusion.

**DEFINITION 4.1.** A (partial) curve diagram  $\Gamma$  is a diagram on  $D_n$  consisting of  $j \leq n - 1$  closed, oriented arcs which are labelled  $\Gamma_1, \dots, \Gamma_j$ . Moreover, the boundary circle of  $D_n$  is labelled  $\Gamma_0$ , and by abuse of notation we shall refer to it as an “arc” of  $\Gamma$ . We require:

- (1) every path component of  $D_n \setminus \Gamma$  has at least one puncture in its interior,
- (2)  $\bigcup_{i=0}^j \text{int}(\Gamma_i)$  is embedded and disjoint from the punctures (where  $\text{int}$  denotes the interior),
- (3) the starting point of the  $i^{\text{th}}$  arc lies in  $\bigcup_{k=0}^{i-1} \Gamma_k$ , i.e. on one of the previous arcs,
- (4) the end point of the  $i^{\text{th}}$  arc lies in one of the previous arcs, or on an earlier point of the  $i^{\text{th}}$  arc, or in a puncture.

In the special case that  $j = n - 1$ , so that in (1) every path component contains precisely one puncture, we say  $\Gamma$  is a total curve diagram.