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ORDERINGS OF MAPPING CLASS GROUPS AFTER THURSTON

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ABSTRACT. We are concerned with mapping class groups of hyperbolic surfaces with nonempty boundary. We present a very natural method, due to Thurston, of finding many different left orderings of such groups. The construction uses the action of the mapping class group on the boundary of the universal cover (viewed in $\mathbf{H}^2$), including its limit points on the circle at infinity. We classify all orderings of braid groups which arise in this way. Moreover, restricting to a certain class of "nonpathological" orderings, we prove that there are only finitely many conjugacy classes of such orderings.

We shall be concerned with certain surfaces S and their mapping class groups $\mathcal{MCG}(S)$. The surfaces under consideration are compact, with a finite set of punctures and nonempty boundary, but not necessarily oriented. We recall that $\mathcal{MCG}(S)$ is the group of isotopy classes of homeomorphisms $S \to S$ which map ∂S identically and permute the punctures. It was first proved by Dehornoy [6] that braid groups (i.e. mapping class groups of punctured disks) are left orderable. A topological proof of this result was given in [9], and the extension to mapping class groups of general surfaces with boundary can be found in [22]. (Note that mapping class groups of surfaces with empty boundary have torsion, and thus cannot be left orderable.) Here we present a very natural method, due to Thurston [24], of finding many different left orderings of such groups. In brief, one equips the surface with a hyperbolic structure, lifts it to $\mathbf{H}^2$, attaches to this cover its limit points on the circle at infinity, and notices that there is a natural action of the mapping class group on the (circular) boundary of the resulting space which fixes a point, and thus an action on $\mathbf{R}$. We classify the set of orderings of braid groups which arise from Thurston's construction (not all orderings do – see the example in 2.6); more precisely, we divide these orderings into two disjoint classes, which we call orderings of finite, respectively infinite, type; the orderings inside each of the classes are classified by combinatorial means. Finite type orderings are discrete, and there exist only finitely many conjugacy classes of them. By