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Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^\infty$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. We are working with polynomials, so that

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) \\ = \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0), \end{aligned}$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right) \\ = \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right). \end{aligned}$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n . \square

4.4 GENERALIZED BERNOULLI POWER SERIES

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n} , where $n \in \mathbf{Z}$, $n \geq 1$, in the field \mathbf{Q}_p , given by

$$(26) \quad B_{-n} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n},$$

where the limit is taken in a p -adic sense. Note that $\phi(p^k) \rightarrow 0$ in \mathbf{Z}_p as $k \rightarrow \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k \rightarrow \infty} \left(1 - p^{\phi(p^k)-n-1} \right) B_{\phi(p^k)-n} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \omega^{-n}) \\ &= nL_p(n + 1; \omega^{-n}). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p,$$

where each prime r is taken to be a rational prime.

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) | n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

Proof. By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) | \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) | n$ if and only if $(p-1) | (\phi(p^k) - n)$. Therefore, if k is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p,$$

and the result follows from (26). \square

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n, \chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field \mathbf{C}_p according to

$$(27) \quad B_{-n, \chi} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n, \chi},$$

where the limit is once again taken in a p -adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m, \chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters χ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n, \chi} &= \lim_{k \rightarrow \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1} \right) B_{\phi(p^k)-n, \chi_{\phi(p^k)}} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \chi_n) \\ &= n L_p(n+1; \chi_n), \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n, 1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $B_{-n, 1} = B_{-n}$ for all such n .

If $k \geq 2$, then $\phi(p^k)$ is even. Thus n and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even.} \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod{2}$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod{2}$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let χ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

Proof. Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k \rightarrow \infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the p -adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the p -adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. \square

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k \rightarrow \infty} (B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p)p^{\phi(p^k)-n-1} B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)) \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n)L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= nL_p(n + 1, t; \chi_n). \end{aligned}$$

Since $L_p(n + 1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such t . Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p(n + m + 1; \chi_{n+m}) \\ &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n + m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \leq \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$(28) \quad B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \geq 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}(mf_\chi) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

Proof. By definition, since $|mf_\chi|_p \leq |q|_p$,

$$\begin{aligned} B_{-n,\chi}(mf_\chi) - B_{-n,\chi}(0) &= \lim_{k \rightarrow \infty} (B_{\phi(p^k)-n,\chi}(mf_\chi) - B_{\phi(p^k)-n,\chi}(0)) \\ &= \lim_{k \rightarrow \infty} (\phi(p^k) - n) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k - 1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a, p) = 1$. These imply that

$$\lim_{k \rightarrow \infty} (\phi(p^k) - n) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. \square

THEOREM 4.16. Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all χ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t).$$

Proof. Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod{2}$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n + \delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t), \end{aligned}$$

and the proof is complete. \square

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