4.4 Generalized Bernoulli power series

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. We are working with polynomials, so that

$$\lim_{i\to\infty}\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) = \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0),$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\lim_{i\to\infty}\left(\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right)$$
$$= \left(\left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right).$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n. $\square$

4.4 GENERALIZED BERNOULLI POWER SERIES

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n}\,, \tag{26}$$

where the limit is taken in a p-adic sense. Note that $\phi(p^k) \to 0$ in $\mathbf{Z}_p$ as $k \to \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned}
B_{-n} &= \lim_{k\to\infty}\left(1 - p^{\phi(p^k)-n-1}\right)B_{\phi(p^k)-n}\\
&= \lim_{k\to\infty} -\left(\phi\left(p^k\right) - n\right)L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \omega^{-n}\right)\\
&= nL_p\left(n+1; \omega^{-n}\right).
\end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. *Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then*

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

where each prime r is taken to be a rational prime.

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) \mid n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

Proof. By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) \mid \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) \mid n$ if and only if $(p-1) \mid (\phi(p^k) - n)$. Therefore, if k is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

and the result follows from (26). $\square$

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{C}_p$ according to

$$B_{-n,\chi} = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi} , \tag{27}$$

where the limit is once again taken in a p-adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m,\chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters χ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n,\chi} &= \lim_{k \to \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n,\chi_{\phi(p^k)}} \\ &= \lim_{k \to \infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \chi_n\right) \\ &= n L_p\left(n+1; \chi_n\right) , \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n,1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $B_{-n,1} = B_{-n}$ for all such n.

If $k \geq 2$, then $\phi(p^k)$ is even. Thus n and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even}. \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let χ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

Proof. Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k\to\infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the p-adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the p-adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k\to\infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p)p^{\phi(p^k)-n-1} B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)\right) \\ &= \lim_{k\to\infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= n L_p(n+1, t; \chi_n). \end{aligned}$$

Since $L_p(n+1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such t. Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p(n+m+1; \chi_{n+m}) \\ &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n+m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \le \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m, \tag{28}$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \ge 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \ge 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \ge 1$. Then for any $m \in \mathbf{Z}$, $m \ge 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

Proof. By definition, since $|mf_\chi|_p \le |q|_p$,

$$\begin{aligned} B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi}\left(mf_\chi\right) - B_{\phi(p^k)-n,\chi}(0)\right) \\ &= \lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k-1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a,p) = 1$. These imply that

$$\lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. $\square$

THEOREM 4.16. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all χ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,*

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t) .$$

Proof. Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m ,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod 2$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n + \delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t) , \end{aligned}$$

and the proof is complete. □

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