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**THEOREM 4.3.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then*

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi).$$

*Proof.* From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1)b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1)c_n(t).$$

From (16), whenever  $n \geq -1$ ,

$$a_n(-t) = \chi(-1)a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi). \quad \square$$

If  $\chi(-1) = -1$  and  $t = 0$ , then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus  $L_p(s; \chi) = 0$  for all  $s \in \mathfrak{D}$ , as we would expect.

#### 4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$ , $\alpha \in \mathbf{C}_p$ , $|\alpha|_p \leq 1$

To develop  $L_p(s, t; \chi)$  in terms of a power series in  $t$  will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

**LEMMA 4.4.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ . Then for  $n \in \mathbf{Z}$ ,  $n \geq 1$ ,*

$$\lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s+n, t; \chi) = -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

*Proof.* Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s-1} + \sum_{m=0}^{\infty} a_m(t)(s-1)^m,$$

where  $a_{-1}(t) = (1 - \chi(p)p^{-1})B_{0, \chi}$ . Thus

$$\lim_{s \rightarrow 1} (s - 1)L_p(s, t; \chi) = (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

Now let  $n \in \mathbf{Z}$ ,  $n \geq 1$ , and consider

$$\lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = \lim_{s \rightarrow 1} \binom{n-s}{n} L_p(s, t; \chi).$$

If  $n = 1$ , then we write this as

$$\lim_{s \rightarrow 1} (1 - s)L_p(s, t; \chi) = - (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

If  $n \geq 2$ , then

$$\frac{1}{n!} \lim_{s \rightarrow 1} \prod_{i=0}^{n-2} (n - s - i) = \frac{1}{n},$$

which implies that

$$\begin{aligned} \lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s + n, t; \chi) &= \frac{1}{n!} \left( \lim_{s \rightarrow 1} \prod_{i=0}^{n-2} (n - s - i) \right) \left( \lim_{s \rightarrow 1} (1 - s)L_p(s, t; \chi) \right) \\ &= -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}. \end{aligned}$$

Therefore the lemma holds for all  $n \geq 1$ .  $\square$

Now, because  $L_p(s, t; 1)$  is undefined when  $s = 1$ , the quantity

$$\binom{-s}{n} L_p(s + n, t; 1)$$

is undefined when  $s = 1 - n$ , for  $n \in \mathbf{Z}$ ,  $n \geq 1$ . However, Lemma 4.4 shows that this quantity exists as  $s \rightarrow 1 - n$ . In the following we will encounter expressions that involve  $\binom{-s}{n} L_p(s + n, t; \chi)$ , and because of Lemma 4.4 we shall assume the understanding that

$$\binom{-s}{n} L_p(s + n, t; \chi) \Big|_{s=1-n} = -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}$$

for  $n \in \mathbf{Z}$ ,  $n \geq 1$ .

**THEOREM 4.5.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then*

$$(21) \quad L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m t^m L_p(s + m; \chi_m).$$

*Proof.* Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and let  $k \in \mathbf{Z}$ ,  $k \geq 1$ . Then

$$\sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-k+m; \chi_m) = -\frac{1}{k} q^k t^k (1 - \chi_k(p)p^{-1}) B_{0, \chi_k} + \sum_{m=0}^{k-1} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m).$$

By evaluating the  $L$ -function, we obtain

$$\binom{k-1}{m} L_p(1-(k-m); \chi_m) = -\frac{1}{k} \binom{k}{m} (1 - \chi_k(p)p^{k-m-1}) B_{k-m, \chi_k},$$

and thus

$$\sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m) = -\frac{1}{k} \sum_{m=0}^k \binom{k}{m} q^m t^m (1 - \chi_k(p)p^{k-m-1}) B_{k-m, \chi_k},$$

which implies that the sum converges for  $s = 1 - k$ . Breaking this into two sums

$$\begin{aligned} \sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m) &= -\frac{1}{k} \sum_{m=0}^k \binom{k}{m} B_{k-m, \chi_k} q^m t^m + \frac{1}{k} \chi_k(p) p^{k-1} \sum_{m=0}^k \binom{k}{m} B_{k-m, \chi_k} p^{-m} q^m t^m \\ &= -\frac{1}{k} (B_{k, \chi_k}(qt) - \chi_k(p) p^{k-1} B_{k, \chi_k}(p^{-1}qt)) \\ &= L_p(1-k, t; \chi). \end{aligned}$$

Thus (21) holds for a sequence  $\{1 - k\}_{k=1}^{\infty}$  that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all  $s$  in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in  $s$ , of each of the functions on either side of (21) contain  $\mathfrak{D}$ , except  $s \neq 1$  when  $\chi = 1$ .

This is obvious for the function  $L_p(s, t; \chi)$ . Consider the function

$$\sum_{m=0}^{\infty} \binom{-s}{m} q^m t^m L_p(s+m; \chi_m) = \sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-s}{m} q^m t^m a_{n, \chi_m} (s+m-1)^n.$$

We have seen that this sum converges for  $s = 1 - k$ , where  $k \in \mathbf{Z}$ ,  $k \geq 1$ . Now we need to show that it converges for  $s = \xi$ , where  $\xi \in \mathfrak{D}$ ,  $\xi \neq 1$  if  $\chi = 1$ , and  $\xi \neq 1 - k$  for  $k \in \mathbf{Z}$ ,  $k \geq 1$ . So let  $\xi$  satisfy these restrictions,

and let  $\epsilon > 0$ . Note that  $|\xi - 1|_p < r$ , where  $r = |p|_p^{1/(p-1)}|q|_p^{-1}$ . Let  $r_0 \in \mathbf{R}$ ,  $0 \leq r_0 < r$ , such that  $|\xi - 1|_p = r_0$ . Then for any  $m \in \mathbf{Z}$ ,  $m \geq 0$ ,

$$\begin{aligned} |\xi + m - 1|_p &\leq \max \left\{ |m|_p, |\xi - 1|_p \right\} \\ &\leq \max \{1, r_0\}, \end{aligned}$$

implying that  $\xi + m \in \mathfrak{D}$ ,  $\xi + m \neq 1$ . Let  $\delta \in \mathbf{R}$  such that  $r^\delta = \max\{1, r_0\}$ . Then  $0 \leq \delta < 1$ , and

$$(22) \quad |\xi + m - 1|_p \leq r^\delta.$$

Let  $N_1 \in \mathbf{Z}$  such that

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any  $m \in \mathbf{Z}$ ,  $m \geq 1$ , such that  $m \geq N_1$ , we must also have

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For  $m \in \mathbf{Z}$ ,  $m \geq 1$ , consider

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p|_p^{-1} |q|_p^m \left| \binom{-\xi}{m} (\xi + m - 1)^{-1} \right|_p.$$

Note that, by (22),

$$\begin{aligned} \left| \binom{-\xi}{m} (\xi + m - 1)^{-1} \right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^m \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\leq |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p|_p^{-1} |q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \geq |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if  $m \geq N_1$ , then

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p < \epsilon.$$

Now let  $N_2 \in \mathbf{Z}$  such that

$$|f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)N_2/(p-1)} |q|_p^{(1-\delta)N_2} < \epsilon.$$

Then we must also have

$$|f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} < \epsilon$$

for any  $m, n \in \mathbf{Z}$  such that  $m \geq 0$ ,  $n \geq 0$ , and  $\max\{m, n\} \geq N_2$ . Let us consider

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq \left| \binom{-\xi}{m} \right|_p |q|_p^m |a_{n, \chi_m}|_p |\xi + m - 1|_p^n,$$

where  $m, n \in \mathbf{Z}$ ,  $m \geq 0$ ,  $n \geq 0$ . For all  $m \geq 0$ ,

$$\left| \binom{-\xi}{m} \right|_p \leq |m!|_p^{-1} r^{\delta m},$$

and by utilizing this along with (17) and (22), our expression becomes

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq |m!(n+1)!|_p^{-1} |f_{\chi} p|_p^{-1} r^{\delta(m+n)} |q|_p^{m+n}.$$

Since

$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$

we obtain

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq |f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)}.$$

Thus if  $\max\{m, n\} \geq N_2$ , then

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p < \epsilon.$$

Let  $N = \max\{N_1, N_2\}$ , and let  $m, n \in \mathbf{Z}$ ,  $m \geq 0$ ,  $n \geq -1$ . Then for  $\max\{m, n\} \geq N$ , it must be true that

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p < \epsilon.$$

Thus, by Proposition 2.4, the sum

$$\sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n$$

must converge. This implies that the function on the right of (21) must converge for all  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ , and the theorem must then hold.  $\square$

Since we can now express  $L_p(s, t; \chi)$  in terms of a power series in  $t$ , we can take a derivative of this function with respect to  $t$ .

LEMMA 4.6. Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ .  
Then

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n!q^n \binom{-s}{n} L_p(s + n, t; \chi_n),$$

for  $n \in \mathbf{Z}$ ,  $n \geq 0$ .

*Proof.* If  $n = 0$ , then the lemma is obviously true. So consider  $n = 1$ . Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s, t; \chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s + m; \chi_m).$$

Now,

$$m \binom{-s}{m} = -s \binom{-s-1}{m-1},$$

so that

$$\begin{aligned} \frac{\partial}{\partial t} L_p(s, t; \chi) &= \sum_{m=1}^{\infty} (-s) \binom{-s-1}{m-1} q^m t^{m-1} L_p(s + m; \chi_m) \\ &= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p(s + 1 + m; \chi_{1+m}) \\ &= -qs L_p(s + 1, t; \chi_1). \end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n!q^n \binom{-s}{n} L_p(s + n, t; \chi_n)$$

for some  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Then

$$\begin{aligned} \frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) &= \frac{\partial}{\partial t} \left( \frac{\partial^n}{\partial t^n} L_p(s, t; \chi) \right) \\ &= n!q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s + n, t; \chi_n). \end{aligned}$$

From the case for  $n = 1$ , we see that

$$\begin{aligned} n!q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s + n, t; \chi_n) &= n!q^n \binom{-s}{n} (-s - n)q L_p(s + n + 1, t; \chi_{n+1}) \\ &= (n + 1)!q^{n+1} \binom{-s}{n + 1} L_p(s + n + 1, t; \chi_{n+1}). \end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) = (n + 1)!q^{n+1} \binom{-s}{n + 1} L_p(s + n + 1, t; \chi_{n+1}),$$

and the lemma must hold by induction.  $\square$

With this result, we can derive a more general power series expansion of  $L_p(s, t; \chi)$ .

**THEOREM 4.7.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then for  $\alpha \in \mathbf{C}_p$ ,  $|\alpha|_p \leq 1$ ,*

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t - \alpha)^m L_p(s + m, \alpha; \chi_m).$$

**REMARK.** Note that Theorem 4.5 is the case of  $\alpha = 0$  here.

*Proof.* It follows from the Taylor series expansion of  $L_p(s, t; \chi)$  in the variable  $t$  about  $\alpha$  (see Proposition 2.6) that we can write  $L_p(s, t; \chi)$  in the form

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \beta_m (t - \alpha)^m,$$

where

$$\beta_m = \frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) \Big|_{t=\alpha}.$$

From Lemma 4.6

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) = \binom{-s}{m} q^m L_p(s + m, t; \chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s + m, \alpha; \chi_m),$$

completing the proof.  $\square$

#### 4.3 RELATING $L_p(s, t; \chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of  $L_p(s, t; \chi)$ .

For the character  $\chi$ , let  $F_0 = \text{lcm}(f_\chi, q)$ . Then  $f_{\chi_n} \mid F_0$  for each  $n \in \mathbf{Z}$ . Also, let  $F$  be a positive multiple of  $pq^{-1}F_0$ .