

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
Autor: BAUM, Paul / CONNES, Alain
Kapitel: 8. TWISTING BY A 2-COCYCLE
DOI: <https://doi.org/10.5169/seals-64793>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where $\mathbf{L}(M)$ is the total \mathbf{L} class of TM and a is any element in $H^*(BG; \mathbf{Q})$.

Kasparov [19] and Mischenko-Fomenko [21] [22] define a map

$$K_0(BG) \rightarrow K_0 C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0 C^*G \\ & \searrow & \swarrow \\ & K^0(\cdot, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of K -homology given in [9] is being used.) \square

COROLLARY 5. *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30].*

For the same reason our conjecture implies the stable¹⁾ form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the C^* -algebra of the projective regular representation of a discrete group.

All of §2 adapts to the projective situation where together with the G -manifold X one is given a 2-cocycle $\gamma \in Z^2(X \rtimes G, S^1)$. For simplicity we

¹⁾ Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J. Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture. Also, Max Karoubi and I have proved that the usual (i.e. complex K -theory) form of Baum-Connes implies the real form of Baum-Connes.

shall stick to the case $X = \text{pt} = \cdot$ and $G \text{ discrete} = \Gamma$; then $\gamma \in Z^2(\Gamma, S^1)$ is a map: $\Gamma \times \Gamma \rightarrow S^1$ such that:

$$\gamma(g_2, g_3) \gamma(g_1 g_2, g_3)^{-1} \gamma(g_1, g_2 g_3) \gamma(g_1, g_2)^{-1} = 1 \quad \text{for every } g_1, g_2, g_3 \in \Gamma.$$

Given a *proper* Γ -manifold Z , a (Γ, γ) -vector bundle on Z is a smooth (complex) vector bundle E on Z together with a smooth map $E \times \Gamma \rightarrow E$ such that (with $\pi: E \rightarrow Z$ the projection):

- a) $\pi(\xi g) = \pi(\xi)g$ for each $\xi \in E$, $g \in \Gamma$;
- b) $\xi(g_1 g_2) = \gamma(g_1, g_2) (\xi g_1) g_2$ for each $g_1, g_2 \in \Gamma$.

In b), $\gamma(g_1, g_2) \in S^1$ is viewed as a complex number of modulus 1. As in §2, we let $V_{(\Gamma, \gamma)}^0(Z)$ be the collection of triples (E_0, E_1, σ) where E_0, E_1 are (Γ, γ) -vector bundles over Z and σ is a smooth morphism of vector bundles such that:

- 1) $\sigma(\xi g) = \sigma(\xi)g$ for each $\xi \in E_0$, $g \in \Gamma$;
- 2) $\text{Support}(\sigma)$ is Γ -compact.

The groups $K_{(\Gamma, \gamma)}^i(Z)$ are then defined as in [5], [31]. The Thom isomorphism as formulated in §2 still holds in this context, and this allows us to define Gysin maps:

$$h!: K_{(\Gamma, \gamma)}^i(T^*Z_1) \rightarrow K_{(\Gamma, \gamma)}^i(T^*Z_2)$$

for a Γ -map h of the proper Γ -manifold Z_1 to the proper Γ -manifold Z_2 .

Thus as in §2 we can define the geometric group also in this twisted situation, we denote it by $K_\gamma^*(X, G)$ in general, and $= K_\gamma^*(\cdot, \Gamma)$ in our special case.

Let then $C_r^*(\Gamma, \gamma)$ be the reduced C^* -algebra of the pair (Γ, γ) , i.e. the C^* -algebra generated in $\ell^2(\Gamma)$ by the projective regular representation λ of Γ :

$$(\lambda(g) \xi)(g') = \gamma(g, g^{-1} g') \xi(g^{-1} g').$$

As in §2 we get a map μ from $K_\gamma^*(\text{pt}, \Gamma)$ to $K_*(C_r^*(\Gamma, \gamma))$, where $\mu(Z, \xi)$ is the analytical index of the K -cocycle $(Z, \xi) \in V_{(\Gamma, \gamma)}^*(T^*Z)$. The only part of the construction which is modified by the presence of γ is that of the C^* -module over $C_r^*(\Gamma, \gamma)$ attached to a (Γ, γ) -bundle E on the proper Γ -manifold Z . More precisely, one starts with the space $C_c(Z, E \otimes \Omega^{1/2})$ of compactly supported continuous $\frac{1}{2}$ -density sections of E and, after choosing a Γ -invariant metric on E , one defines:

$$\langle \xi, \eta \rangle(g) = \int_X \langle \xi_x, (\eta_{xg}) g^{-1} \rangle \quad \text{for each } g \in \Gamma,$$

which gives a $C_c(\Gamma)$ -valued sesquilinear form on $C_c(Z, E \otimes \Omega^{1/2})$. One checks that for any $\xi \in C_c(Z, E \otimes \Omega^{1/2})$, $\langle \xi, \xi \rangle$ is a *positive* element of $C_r^*(\Gamma)$, since for any $\eta \in \ell^2(\Gamma)$ one has:

$$\begin{aligned} \langle \eta, \lambda(\langle \xi, \xi \rangle) \eta \rangle &= \sum \bar{\eta}(g) \langle \xi, \xi \rangle(h) (\lambda(h) \eta)(g) \\ &= \sum \gamma(h, h^{-1}g) \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle \xi_x, (\xi_x h) h^{-1} \rangle \\ &= \sum \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle (\xi_{xg^{-1}})g, (\xi_{xg^{-1}h})h^{-1}g \rangle \geq 0. \end{aligned}$$

Then, by completion with respect to the norm $\|\langle \xi, \xi \rangle\|^{1/2}$, one gets a C^* -module over $C_r^*(\Gamma, \gamma)$, which we denote by $L^2(Z, E)$. The right action is given by:

$$(\xi f)(x) = \sum_{\Gamma} (\xi_{xg^{-1}})g f(g) \text{ for each } \xi \in C_c(Z, E \otimes \Omega^{1/2}), f \in C_c(\Gamma).$$

Next, we can choose a Γ -invariant Riemannian metric on Z , represent every class in $K_{(\Gamma, \gamma)}^0(T^*Z)$ by a pair E_0, E_1 of (Γ, γ) -hermitian bundles on Z and a symbol σ which is an isomorphism of the pull back of E_0 to S^*Z to that of E_1 , and is independent of ξ , $\pi(\xi) = z$, outside a Γ -compact subset of Z . Letting P_σ be the corresponding order 0 pseudo-differential operator, one gets a Kasparov $(\mathbb{C}, C_r^*(\Gamma, \gamma))$ -bimodule: the triple $(L^2(Z, E_0), L^2(Z, E_1), P_\sigma)$ which gives an element of $K_0(C_r^*(\Gamma, \gamma))$. It is important to give another description of the map $\mu: K_{(\Gamma, \gamma)}^0(T^*Z) \rightarrow K_0(C_r^*(\Gamma, \gamma))$, using Kasparov products.

PROPOSITION 1. a) *Let X be a proper Γ -manifold, then $K_{(\Gamma, \gamma)}^i(X)$ is canonically isomorphic to $K_i(C_0(X) \rtimes_\gamma \Gamma)$, where $C_0(X) \rtimes_\gamma \Gamma$ is the twisted crossed product of $C_0(X)$ by Γ .*

b) (Compare [19]). *For any C^* -algebras A, B on which Γ acts by automorphisms, one has a natural map from $KK_\Gamma(A, B)$ to $KK(A \rtimes_\gamma \Gamma, B \rtimes_\gamma \Gamma)$.*

Proof. a) One can consider $A = C_0(X) \rtimes_\gamma \Gamma$ as the C^* -algebra of the groupoid $X \rtimes \Gamma = G$ with units $G^{(0)} = X$, source and range maps $s(x, g) = xg$, $r(x, g) = x$ and composition $(x, g) \cdot (x', g') = (x, gg')$ with the 2-cocycle $\gamma \circ \pi$ where π is the natural homomorphism $G \rightarrow \Gamma: \pi(x, g) = g$.

Thus A is the completion of this convolution algebra $C_c(G)$:

$$\begin{aligned} (f_1 * f_2)(x, g) &= \sum_{\Gamma} f_1(x, h) f_2(xh, h^{-1}g) \gamma(h, h^{-1}g) \\ f^*(x, g) &= \bar{f}(xg, g^{-1}) \end{aligned}$$

with the norm $\|f\| = \sup \|\pi_x(f)\|$, where for each $x \in X$ the representation π_x of $C_c(G)$ in $\ell^2(\Gamma)$ is given by:

$$(\pi_x(f)\xi)(g) = \sum_{\Gamma} f(xg^{-1}, h) \xi(h^{-1}g) \gamma(h, h^{-1}g) \text{ for each } \xi \in \ell^2(\Gamma).$$

Now, given a (Γ, γ) -vector bundle E on X , one can endow E with a Γ -invariant hermitian metric and define a C^* -module \mathcal{E} over $A = C_0(X) \rtimes_{\gamma} \Gamma$ as follows. For any $\xi, \eta \in C_c(X, E)$ let $\langle \xi, \eta \rangle \in C_c(X \rtimes \Gamma)$ be given by $\langle \xi, \eta \rangle(x, g) = \langle \xi_x g, \eta_{xg} \rangle$; then $\langle \xi, \xi \rangle$ is a positive element of $A = C_0(X) \rtimes_{\gamma} \Gamma$, since for any $\eta \in \ell^2(\Gamma)$ and $x \in X$ one has:

$$\begin{aligned} \langle \eta, \pi_x(\langle \xi, \xi \rangle) \eta \rangle = \\ \sum \sum \langle \xi_{xg^{-1}} h, \xi_{xg^{-1}} h \rangle \eta(h^{-1}g) \bar{\eta}(g) \gamma(h, h^{-1}g) = \langle \alpha, \alpha \rangle \geq 0, \end{aligned}$$

where $\alpha = \sum (\xi_{xg^{-1}})g \eta(g) \in E_x$.

Let \mathcal{E} be the completion of $C_c(X, E)$ with the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|$; then \mathcal{E} is a C^* -module over A , with:

$$(\xi f)(x) = \sum f(xg^{-1}, g) \xi(xg')g \text{ for every } f \in C_c(X \rtimes \Gamma), \xi \in C_0(X, E).$$

(One easily checks that $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle * f$ and that this right action of $C_c(X \rtimes \Gamma)$ extends to an action of A .)

The equality $(\eta \langle \eta, \xi \rangle)(x) = \sum \langle (\eta_{xg^{-1}})g, \xi_x \rangle (\eta_{xg^{-1}})g$ shows that any endomorphism σ of the vector bundle E which commutes with Γ and has Γ -compact support defines an A -compact endomorphism of \mathcal{E} by the equality: $(T\xi)(x) = \sigma(x) \xi(x)$ for every $x \in X$. Thus, to any triple $(E_0, E_1, \sigma) \in V_{(\Gamma, \gamma)}^0(X)$ corresponds an element of $KK(\mathbf{C}, A)$, $A = C_0(X) \rtimes_{\gamma} \Gamma$, which obviously depends only upon the class of the triple in $K_{(\Gamma, \gamma)}^0(X)$. Let us prove that this map is an isomorphism assuming that Γ is *torsion free*. We may then assume that X is Γ -compact. We claim first that $A = C_0(X) \rtimes_{\gamma} \Gamma$ is Morita equivalent to a C^* -algebra with unit. Indeed, with $V = X/\Gamma$, A is the C^* -algebra of the continuous field of elementary C^* -algebras $A_t = C_0(\pi^{-1}(t)) \rtimes_{\gamma} \Gamma$, where $\pi: X \rightarrow X/\Gamma = V$ is the projection. By a simple computation, one gets that the Dixmier-Douady obstruction $\delta(A) \in H^3(V, \mathbf{Z})$ is given by $\delta(A) = \phi^*(\partial\gamma)$ where $\phi: V \rightarrow B\Gamma$ is the classifying map, and $\partial\gamma \in H^3(B\Gamma, \mathbf{Z})$ is the boundary of $\gamma \in H^2(B\Gamma, S^1) = H^2(\Gamma, S^1)$ in the exact sequence:

$$H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \xrightarrow{\partial} H^3(\Gamma, \mathbf{Z}) \rightarrow H^3(\Gamma, \mathbf{R}) \rightarrow \dots$$

In particular $\delta(A)$ is a torsion element in $H^3(V, \mathbf{Z})$ so that there exists a bundle of matrix algebras over V with the same Dixmier-Douady obstruction and A is Morita equivalent to a unital C^* -algebra. It follows then that $K_0(A)$

is obtained from C^* -modules \mathcal{E} over A with the property $\text{id}_{\mathcal{E}} \in \text{End}_A^0(\mathcal{E})$, i.e. all endomorphisms of \mathcal{E} are A -compact. Finally, the above construction sets up a surjective map from (Γ, γ) -vector bundles on X to C^* -modules over A with the above property. Given \mathcal{E} , the fiber E_x of the corresponding vector bundle is:

$$E_x = \mathcal{E} \widehat{\otimes}_A \ell^2(\Gamma)$$

where $A = C_0(X) \rtimes_{\gamma} \Gamma$ acts in $\ell^2(\Gamma)$ by the representation π_x . Since $\pi_x(A) \subset \text{Compacts}$, one gets that E_x is a finite dimensional Hilbert space.

b) The proof is the same as in [19], one defines for any Γ -equivariant C^* -module \mathcal{E} over B the crossed product $\mathcal{E} \rtimes_{\gamma} \Gamma$ twisted by the 2-cocycle γ . \square

We can now state:

THEOREM 2. *For any element x of $K_{(\Gamma, \gamma)}^0(T^*Z) = K_0(A)$ (where $A = C_0(T^*Z) \rtimes_{\gamma} \Gamma$, and Z a proper Γ -manifold), one has:*

$$\mu(x) = x \otimes j_{(\Gamma, \gamma)}(D),$$

where $D \in KK_{\Gamma}(C_0(T^*Z), \mathbf{C})$ is the class of the Dirac operator.

Note that $x \in KK(\mathbf{C}, C_0(T^*Z) \rtimes_{\gamma} \Gamma)$ and that

$$j_{(\Gamma, \gamma)}(D) \in KK(C_0(T^*Z) \rtimes_{\gamma} \Gamma, C_r^*(\Gamma, \gamma)),$$

so that the above equality is meaningful. The proof is straightforward.

To show how to use this theorem, we shall combine it with the recent result of G. G. Kasparov ([19]) to compute $K_i(C_r^*(\Gamma, \gamma))$ in the following example: we let $\Gamma = \pi_1(M)$ be the fundamental group of a Riemann surface M with genus > 1 . From the exact sequence $0 \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow 0$ one gets $H^2(\Gamma, S^1) = \mathbf{R}/\mathbf{Z}$, so that there are many non trivial cocycles in this example. The geometric group $K_{\gamma}^i(\text{pt}, \Gamma)$ is easily determined: since the universal cover \tilde{M} of M (the Poincaré disc) is a final object in the category of proper Γ -manifolds, and homotopy classes of Γ -maps, it is enough to compute $K_{(\Gamma, \gamma)}^i(T^*\tilde{M})$. Since \tilde{M} has a Γ -invariant Spin^c -structure, the Thom isomorphism hence gives: $K_{\gamma}^i(\text{pt}, \Gamma) = K_{(\Gamma, \gamma)}^i(\tilde{M})$. By Proposition 1, one has $K_{(\Gamma, \gamma)}^i(\tilde{M}) = K_i(C_0(\tilde{M}) \rtimes_{\gamma} \Gamma)$ and the latter C^* -algebra is Morita equivalent to $C(M)$ (see the proof of a) in Proposition 1). Thus we get: $K_{\gamma}^0(\text{pt}, \Gamma) = \mathbf{Z}^2$, $K_{\gamma}^1(\text{pt}, \Gamma) = \mathbf{Z}^{2g}$.

THEOREM 3. *Let Γ be the fundamental group of a Riemann surface of genus > 1 , and $\gamma \in H^2(\Gamma, S^1)$, then the map $\mu: K_\gamma^*(\text{pt}, \Gamma) \rightarrow K_*(C_r^*(\Gamma, \gamma))$ is an isomorphism.*

Proof. Let $D \in KK_G(C_0(U), \mathbf{C})$ be the $G = \text{PSL}(2, \mathbf{R})$ equivariant Dirac operator on the Poincaré disc $U = G/G_c$ (cf. [19]). Identify \tilde{M} with U and Γ with a subgroup of G . Then by Proposition 1 b) and Theorem 2 it is enough to show that the restriction of D to an element of $KK_\Gamma(C_0(U), \mathbf{C})$ is an invertible element. This follows from [19] which shows that D is an invertible element of $KK_G(C_0(U), \mathbf{C})$, and from the multiplicative property of the restriction to subgroups.

We shall now show how to prove that the C^* -algebras $C_r^*(\Gamma, \gamma)$ are pairwise non-isomorphic when γ varies in $H^2(\Gamma, S^1)$. In fact we shall compute in full generality the composition $\zeta \circ \mu$ of the canonical trace ζ on $C_r^*(\Gamma, \gamma)$ (viewed as a map from K_0 to \mathbf{C}) with the above map $\mu: K_\gamma^0(\text{pt}, \Gamma) \rightarrow K_0(C_r^*(\Gamma, \gamma))$.

The computation is a generalization of the index theorem for covering spaces of Atiyah ([3]).

LEMMA 4. *Let Z be a proper Γ -manifold and E a (Γ, γ) vector bundle on Z . There exists a Γ -invariant connection ∇ on E .*

Proof. For any (Γ, γ) -vector bundle F on Z and section $\xi \in C_c^\infty(Z, F)$ let, for $g \in \Gamma$, $g\xi \in C_c^\infty(Z, F)$ be given by: $(g\xi)(x) = (\xi(xg))g^{-1} \in F_x$ for every $x \in Z$.

In this way one gets a natural γ -action of Γ on both $C_c^\infty(Z, E)$ and $C_c^\infty(Z, E \otimes T^*Z)$, and one looks for a connection

$$\nabla: C_c^\infty(Z, E) \rightarrow C_c^\infty(Z, E \otimes T^*Z)$$

such that $\nabla(g\xi) = g(\nabla\xi)$ for every ξ . Let $f \in C^\infty(Z)$, $0 \leq f \leq 1$, be such that $\sum_\Gamma f(xg) = 1$ for every $x \in Z$ and ∇_0 be a connection on E . Put $\nabla = \sum_\Gamma g^{-1}(f\nabla_0)g$. By construction ∇ is Γ -invariant, moreover each $g^{-1}\nabla_0 g$ is a connection on E thus ∇ is a connection on E . \square

Proof of Theorem 3, continued. Assuming now that Z is Γ -compact, let for a Γ -invariant connection ∇ on E , ω_∇ be the canonical differential form on Z which represents locally the Chern character $\text{ch}(E)$. By construction ω_∇ is Γ -invariant and hence determines a cohomology class in Z/Γ . One checks as usual that this class does not depend upon the choice of ∇ and

we shall denote it by $[E] \in H^*(Z/\Gamma, \mathbf{R})$. This construction easily extends to give a map ch from $K_{(\Gamma, \gamma)}^0(Z)$ to $H^*(Z/\Gamma, \mathbf{R})$ for any proper Γ -manifold Z . However, in the presence of the 2-cocycle γ the range of this map is *no longer necessarily contained* in $H^*(Z/\Gamma, \mathbf{Q})$.

To be more precise, let us make a few simplifying assumptions and compute exactly the range of this Chern character:

$$\text{ch}: K_{(\Gamma, \gamma)}^0(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R}).$$

Thus let us assume that Γ is torsion free and that the image of $\gamma \in H^2(\Gamma, S^1)$ in $H^3(\Gamma, \mathbf{Z})$ under the connecting map of the long exact sequence:

$$\dots \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow H^3(\Gamma, \mathbf{Z}) \rightarrow \dots$$

is equal to 0 (it is always a torsion element).

Let then $\rho \in H^2(\Gamma, \mathbf{R})$ be such that $e(\rho) = \gamma$ where $e: \mathbf{R} \rightarrow S^1$ is given by $e(s) = \exp(2\pi i s)$, for each $s \in \mathbf{R}$.

LEMMA 5. a) Let $\rho \in Z^2(\Gamma, \mathbf{R})$ and Z be a proper Γ -manifold, then there exists a smooth function $c \in C^\infty(Z \rtimes \Gamma)$ such that:

$$c(x, g_1) + c(xg_1, g_2) = c(x, g_1g_2) - \rho(g_1, g_2)$$

for every $x \in Z$, $g_1, g_2 \in \Gamma$.

b) If $\gamma = e(\rho)$ there exists an isomorphism $r: K_\Gamma^0(Z) \rightarrow K_{(\Gamma, \gamma)}^0(Z)$ making the following diagram commutative:

$$\begin{array}{ccc} K_\Gamma^0(Z) & \xrightarrow{r} & K_{(\Gamma, \gamma)}^0(Z) \\ \downarrow \text{ch} & & \downarrow \text{ch} \\ H^*(Z/\Gamma) & \xrightarrow{m} & H^*(Z/\Gamma), \end{array}$$

where m is multiplication by the cohomology class $\exp(\phi^*\rho)$ and where $\phi: Z/\Gamma \rightarrow B\Gamma$ is the classifying map.

Proof. a) Let $M = Z/\Gamma$, $\pi: Z \rightarrow M$ the projection. Since Z is a locally trivial Γ -principal bundle, it is easy to construct c on the open set $\pi^{-1}(U)$ for U small enough. Then one combines such c_U by a smooth partition of unity on M :

$$c(x, g) = \sum \phi_U(\pi(x)) c_U(x, g).$$

b) Let $c \in C^\infty(Z \rtimes \Gamma)$ be as in a) and let us endow the trivial line bundle on Z (with total space $Z \times \mathbf{C}$) with a structure of (Γ, γ) -bundle. We take:

$$(x, \lambda)g = (xg, e(c(x, g))\lambda).$$

(One has $((x, \lambda)g_1)g_2 = (xg_1g_2, e(c(x, g_1) + c(xg_1, g_2))\lambda) = \gamma^{-1}(g_1, g_2)(x\lambda)(g_1g_2)$.)

Let L be the (Γ, γ) -line bundle on Z thus obtained. It is obvious that tensoring by L gives an isomorphism of $V_{(\Gamma)}^0(Z)$ with $V_{(\Gamma, \gamma)}^0 Z$ and hence of $K_{\Gamma}^0(Z)$ with $K_{(\Gamma, \gamma)}^0(Z)$. \square

End of proof of Theorem 3. To conclude, it is enough to compute $\text{ch}(L)$. Let $\xi \in C^\infty(Z, L)$ be the section $\xi(x) = 1$ for every $x \in Z$. Let ∇ be a Γ -invariant connection on L , one has $\text{ch}(L) = \exp(\omega)$ where $\omega \in H^2(Z/\Gamma, \mathbf{R})$ corresponds to the Γ -invariant 2-form $\theta = \frac{1}{2\pi i} d(\nabla \xi / \xi)$ on Z . Let $\alpha = \frac{1}{2\pi i} \nabla \xi / \xi$, then α is a 1-form on Z , and let us compute for any $g \in \Gamma$ the difference $\alpha - \phi^* \alpha$ where $\phi(x) = xg$ for every $x \in Z$. Since ∇ is Γ -invariant, one has $\phi^* \alpha = \frac{1}{2\pi i} \nabla g(\xi) / g(\xi)$, and as $g(\xi)(x) = e(c(xg, g^{-1})) \xi(x)$ one gets $\phi^* \alpha - \alpha = d\psi_g$, where $\psi_g(x) = c(xg, g^{-1})$ for every $x \in Z$. One has $\psi_{g_1g_2} - g_1\psi_{g_2} - \psi_{g_1} = \rho(g_2^{-1}, g_1^{-1})$. This shows that the class of θ in $H^2(Z/\Gamma, \mathbf{R})$ is the pull back of the class of $-\rho$ in $H^2(B\Gamma, \mathbf{R})$, by the classifying map: $Z/\Gamma \rightarrow B\Gamma$. \square

Using this map $\text{ch}: K_{(\Gamma, \gamma)}^*(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R})$ we get, by the same five steps as in §6, a map

$$K_{\gamma}^*(\text{pt}, \Gamma) \xrightarrow{\text{ch}} H_*(B\Gamma, \mathbf{R}).$$

Again as in §6, let ϵ be the map from $B\Gamma$ to a point, and tr_{Γ} be the canonical trace on $C_r^*(\Gamma, \gamma)$.

THEOREM 6. *For any discrete group Γ and 2-cocycle γ the following diagram is commutative:*

$$\begin{array}{ccc} K_{\gamma}^0(\text{pt}, \Gamma) & \xrightarrow{\mu} & K_0(C_r^*(\Gamma, \gamma)) \\ \downarrow \text{ch} & & \downarrow \text{tr}_{\Gamma} \\ H_*(B\Gamma, \mathbf{R}) & \xrightarrow{\epsilon^*} & \mathbf{C}. \end{array}$$

The proof is a simple adaptation of the heat equation method to compute the Γ -index of the (Γ, γ) -Dirac operator on a Γ -manifold Z .

COROLLARY 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of \mathbf{R} , $\Delta = \text{tr}_\Gamma(K_0(C_r^*(\Gamma, \gamma)))$ contains the group:*

$$\langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle.$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma, \mathbf{R})/H^2(\Gamma, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C_r^*(\Gamma, \gamma))$ by the canonical trace $\zeta = \text{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma, \gamma)$ (for any value of γ), one gets that the corresponding C^* -algebras are isomorphic only when the Γ 's are the same (using K_1) and when the γ 's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞ -manifold, and let F be a C^∞ -foliation of V . Thus F is a C^∞ -integrable sub-vector bundle of TV . As in [33] let G be the holonomy groupoid (graph) of (V, F) . The manifold V is assumed to be Hausdorff and second countable. G , however, is a C^∞ -manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞ -paths

$$\gamma: [0, 1] \rightarrow V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0, 1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{r} \end{smallmatrix} V$.

Let Z be a possibly non-Hausdorff C^∞ -manifold. Assume given a C^∞ -map $\rho: Z \rightarrow V$, set

$$Z \circ G = \{(z, \gamma) \in Z \times G \mid \rho(z) = s(\gamma)\}.$$

A C^∞ right action of G on Z is a C^∞ -map