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The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where $\mathbf{L}(M)$ is the total \mathbf{L} class of TM and a is any element in $H^*(BG; \mathbf{Q})$.

Kasparov [19] and Mischenko-Fomenko [21] [22] define a map

$$K_0(BG) \rightarrow K_0 C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0 C^*G \\ & \searrow & \swarrow \\ & K^0(\cdot, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of K -homology given in [9] is being used.) \square

COROLLARY 5. *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30].*

For the same reason our conjecture implies the stable¹⁾ form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the C^* -algebra of the projective regular representation of a discrete group.

All of §2 adapts to the projective situation where together with the G -manifold X one is given a 2-cocycle $\gamma \in Z^2(X \rtimes G, S^1)$. For simplicity we

¹⁾ Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J. Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture. Also, Max Karoubi and I have proved that the usual (i.e. complex K -theory) form of Baum-Connes implies the real form of Baum-Connes.

shall stick to the case $X = \text{pt} = \cdot$ and $G \text{ discrete} = \Gamma$; then $\gamma \in Z^2(\Gamma, S^1)$ is a map: $\Gamma \times \Gamma \rightarrow S^1$ such that:

$$\gamma(g_2, g_3) \gamma(g_1 g_2, g_3)^{-1} \gamma(g_1, g_2 g_3) \gamma(g_1, g_2)^{-1} = 1 \quad \text{for every } g_1, g_2, g_3 \in \Gamma.$$

Given a *proper* Γ -manifold Z , a (Γ, γ) -vector bundle on Z is a smooth (complex) vector bundle E on Z together with a smooth map $E \times \Gamma \rightarrow E$ such that (with $\pi: E \rightarrow Z$ the projection):

- a) $\pi(\xi g) = \pi(\xi)g$ for each $\xi \in E$, $g \in \Gamma$;
- b) $\xi(g_1 g_2) = \gamma(g_1, g_2) (\xi g_1) g_2$ for each $g_1, g_2 \in \Gamma$.

In b), $\gamma(g_1, g_2) \in S^1$ is viewed as a complex number of modulus 1. As in §2, we let $V_{(\Gamma, \gamma)}^0(Z)$ be the collection of triples (E_0, E_1, σ) where E_0, E_1 are (Γ, γ) -vector bundles over Z and σ is a smooth morphism of vector bundles such that:

- 1) $\sigma(\xi g) = \sigma(\xi)g$ for each $\xi \in E_0$, $g \in \Gamma$;
- 2) $\text{Support}(\sigma)$ is Γ -compact.

The groups $K_{(\Gamma, \gamma)}^i(Z)$ are then defined as in [5], [31]. The Thom isomorphism as formulated in §2 still holds in this context, and this allows us to define Gysin maps:

$$h!: K_{(\Gamma, \gamma)}^i(T^*Z_1) \rightarrow K_{(\Gamma, \gamma)}^i(T^*Z_2)$$

for a Γ -map h of the proper Γ -manifold Z_1 to the proper Γ -manifold Z_2 .

Thus as in §2 we can define the geometric group also in this twisted situation, we denote it by $K_\gamma^*(X, G)$ in general, and $= K_\gamma^*(\cdot, \Gamma)$ in our special case.

Let then $C_r^*(\Gamma, \gamma)$ be the reduced C^* -algebra of the pair (Γ, γ) , i.e. the C^* -algebra generated in $\ell^2(\Gamma)$ by the projective regular representation λ of Γ :

$$(\lambda(g) \xi)(g') = \gamma(g, g^{-1} g') \xi(g^{-1} g').$$

As in §2 we get a map μ from $K_\gamma^*(\text{pt}, \Gamma)$ to $K_*(C_r^*(\Gamma, \gamma))$, where $\mu(Z, \xi)$ is the analytical index of the K -cocycle $(Z, \xi) \in V_{(\Gamma, \gamma)}^*(T^*Z)$. The only part of the construction which is modified by the presence of γ is that of the C^* -module over $C_r^*(\Gamma, \gamma)$ attached to a (Γ, γ) -bundle E on the proper Γ -manifold Z . More precisely, one starts with the space $C_c(Z, E \otimes \Omega^{1/2})$ of compactly supported continuous $\frac{1}{2}$ -density sections of E and, after choosing a Γ -invariant metric on E , one defines:

$$\langle \xi, \eta \rangle(g) = \int_X \langle \xi_x, (\eta_{xg}) g^{-1} \rangle \quad \text{for each } g \in \Gamma,$$

which gives a $C_c(\Gamma)$ -valued sesquilinear form on $C_c(Z, E \otimes \Omega^{1/2})$. One checks that for any $\xi \in C_c(Z, E \otimes \Omega^{1/2})$, $\langle \xi, \xi \rangle$ is a *positive* element of $C_r^*(\Gamma)$, since for any $\eta \in \ell^2(\Gamma)$ one has:

$$\begin{aligned} \langle \eta, \lambda(\langle \xi, \xi \rangle) \eta \rangle &= \sum \bar{\eta}(g) \langle \xi, \xi \rangle(h) (\lambda(h) \eta)(g) \\ &= \sum \gamma(h, h^{-1}g) \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle \xi_x, (\xi_x h) h^{-1} \rangle \\ &= \sum \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle (\xi_{xg^{-1}})g, (\xi_{xg^{-1}h})h^{-1}g \rangle \geq 0. \end{aligned}$$

Then, by completion with respect to the norm $\|\langle \xi, \xi \rangle\|^{1/2}$, one gets a C^* -module over $C_r^*(\Gamma, \gamma)$, which we denote by $L^2(Z, E)$. The right action is given by:

$$(\xi f)(x) = \sum_{\Gamma} (\xi_{xg^{-1}})g f(g) \text{ for each } \xi \in C_c(Z, E \otimes \Omega^{1/2}), f \in C_c(\Gamma).$$

Next, we can choose a Γ -invariant Riemannian metric on Z , represent every class in $K_{(\Gamma, \gamma)}^0(T^*Z)$ by a pair E_0, E_1 of (Γ, γ) -hermitian bundles on Z and a symbol σ which is an isomorphism of the pull back of E_0 to S^*Z to that of E_1 , and is independent of ξ , $\pi(\xi) = z$, outside a Γ -compact subset of Z . Letting P_σ be the corresponding order 0 pseudo-differential operator, one gets a Kasparov $(\mathbb{C}, C_r^*(\Gamma, \gamma))$ -bimodule: the triple $(L^2(Z, E_0), L^2(Z, E_1), P_\sigma)$ which gives an element of $K_0(C_r^*(\Gamma, \gamma))$. It is important to give another description of the map $\mu: K_{(\Gamma, \gamma)}^0(T^*Z) \rightarrow K_0(C_r^*(\Gamma, \gamma))$, using Kasparov products.

PROPOSITION 1. a) *Let X be a proper Γ -manifold, then $K_{(\Gamma, \gamma)}^i(X)$ is canonically isomorphic to $K_i(C_0(X) \rtimes_\gamma \Gamma)$, where $C_0(X) \rtimes_\gamma \Gamma$ is the twisted crossed product of $C_0(X)$ by Γ .*

b) (Compare [19]). *For any C^* -algebras A, B on which Γ acts by automorphisms, one has a natural map from $KK_\Gamma(A, B)$ to $KK(A \rtimes_\gamma \Gamma, B \rtimes_\gamma \Gamma)$.*

Proof. a) One can consider $A = C_0(X) \rtimes_\gamma \Gamma$ as the C^* -algebra of the groupoid $X \rtimes \Gamma = G$ with units $G^{(0)} = X$, source and range maps $s(x, g) = xg$, $r(x, g) = x$ and composition $(x, g) \cdot (x', g') = (x, gg')$ with the 2-cocycle $\gamma \circ \pi$ where π is the natural homomorphism $G \rightarrow \Gamma: \pi(x, g) = g$.

Thus A is the completion of this convolution algebra $C_c(G)$:

$$\begin{aligned} (f_1 * f_2)(x, g) &= \sum_{\Gamma} f_1(x, h) f_2(xh, h^{-1}g) \gamma(h, h^{-1}g) \\ f^*(x, g) &= \bar{f}(xg, g^{-1}) \end{aligned}$$

with the norm $\|f\| = \sup \|\pi_x(f)\|$, where for each $x \in X$ the representation π_x of $C_c(G)$ in $\ell^2(\Gamma)$ is given by:

$$(\pi_x(f)\xi)(g) = \sum_{\Gamma} f(xg^{-1}, h) \xi(h^{-1}g) \gamma(h, h^{-1}g) \text{ for each } \xi \in \ell^2(\Gamma).$$

Now, given a (Γ, γ) -vector bundle E on X , one can endow E with a Γ -invariant hermitian metric and define a C^* -module \mathcal{E} over $A = C_0(X) \rtimes_{\gamma} \Gamma$ as follows. For any $\xi, \eta \in C_c(X, E)$ let $\langle \xi, \eta \rangle \in C_c(X \rtimes \Gamma)$ be given by $\langle \xi, \eta \rangle(x, g) = \langle \xi_x g, \eta_{xg} \rangle$; then $\langle \xi, \xi \rangle$ is a positive element of $A = C_0(X) \rtimes_{\gamma} \Gamma$, since for any $\eta \in \ell^2(\Gamma)$ and $x \in X$ one has:

$$\begin{aligned} \langle \eta, \pi_x(\langle \xi, \xi \rangle) \eta \rangle = \\ \sum \sum \langle \xi_{xg^{-1}} h, \xi_{xg^{-1}} h \rangle \eta(h^{-1}g) \bar{\eta}(g) \gamma(h, h^{-1}g) = \langle \alpha, \alpha \rangle \geq 0, \end{aligned}$$

where $\alpha = \sum (\xi_{xg^{-1}})g \eta(g) \in E_x$.

Let \mathcal{E} be the completion of $C_c(X, E)$ with the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|$; then \mathcal{E} is a C^* -module over A , with:

$$(\xi f)(x) = \sum f(xg^{-1}, g) \xi(xg')g \text{ for every } f \in C_c(X \rtimes \Gamma), \xi \in C_0(X, E).$$

(One easily checks that $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle * f$ and that this right action of $C_c(X \rtimes \Gamma)$ extends to an action of A .)

The equality $(\eta \langle \eta, \xi \rangle)(x) = \sum \langle (\eta_{xg^{-1}})g, \xi_x \rangle (\eta_{xg^{-1}})g$ shows that any endomorphism σ of the vector bundle E which commutes with Γ and has Γ -compact support defines an A -compact endomorphism of \mathcal{E} by the equality: $(T\xi)(x) = \sigma(x) \xi(x)$ for every $x \in X$. Thus, to any triple $(E_0, E_1, \sigma) \in V_{(\Gamma, \gamma)}^0(X)$ corresponds an element of $KK(\mathbf{C}, A)$, $A = C_0(X) \rtimes_{\gamma} \Gamma$, which obviously depends only upon the class of the triple in $K_{(\Gamma, \gamma)}^0(X)$. Let us prove that this map is an isomorphism assuming that Γ is *torsion free*. We may then assume that X is Γ -compact. We claim first that $A = C_0(X) \rtimes_{\gamma} \Gamma$ is Morita equivalent to a C^* -algebra with unit. Indeed, with $V = X/\Gamma$, A is the C^* -algebra of the continuous field of elementary C^* -algebras $A_t = C_0(\pi^{-1}(t)) \rtimes_{\gamma} \Gamma$, where $\pi: X \rightarrow X/\Gamma = V$ is the projection. By a simple computation, one gets that the Dixmier-Douady obstruction $\delta(A) \in H^3(V, \mathbf{Z})$ is given by $\delta(A) = \phi^*(\partial\gamma)$ where $\phi: V \rightarrow B\Gamma$ is the classifying map, and $\partial\gamma \in H^3(B\Gamma, \mathbf{Z})$ is the boundary of $\gamma \in H^2(B\Gamma, S^1) = H^2(\Gamma, S^1)$ in the exact sequence:

$$H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \xrightarrow{\partial} H^3(\Gamma, \mathbf{Z}) \rightarrow H^3(\Gamma, \mathbf{R}) \rightarrow \dots$$

In particular $\delta(A)$ is a torsion element in $H^3(V, \mathbf{Z})$ so that there exists a bundle of matrix algebras over V with the same Dixmier-Douady obstruction and A is Morita equivalent to a unital C^* -algebra. It follows then that $K_0(A)$

is obtained from C^* -modules \mathcal{E} over A with the property $\text{id}_{\mathcal{E}} \in \text{End}_A^0(\mathcal{E})$, i.e. all endomorphisms of \mathcal{E} are A -compact. Finally, the above construction sets up a surjective map from (Γ, γ) -vector bundles on X to C^* -modules over A with the above property. Given \mathcal{E} , the fiber E_x of the corresponding vector bundle is:

$$E_x = \mathcal{E} \widehat{\otimes}_A \ell^2(\Gamma)$$

where $A = C_0(X) \rtimes_{\gamma} \Gamma$ acts in $\ell^2(\Gamma)$ by the representation π_x . Since $\pi_x(A) \subset \text{Compacts}$, one gets that E_x is a finite dimensional Hilbert space.

b) The proof is the same as in [19], one defines for any Γ -equivariant C^* -module \mathcal{E} over B the crossed product $\mathcal{E} \rtimes_{\gamma} \Gamma$ twisted by the 2-cocycle γ . \square

We can now state:

THEOREM 2. *For any element x of $K_{(\Gamma, \gamma)}^0(T^*Z) = K_0(A)$ (where $A = C_0(T^*Z) \rtimes_{\gamma} \Gamma$, and Z a proper Γ -manifold), one has:*

$$\mu(x) = x \otimes j_{(\Gamma, \gamma)}(D),$$

where $D \in KK_{\Gamma}(C_0(T^*Z), \mathbf{C})$ is the class of the Dirac operator.

Note that $x \in KK(\mathbf{C}, C_0(T^*Z) \rtimes_{\gamma} \Gamma)$ and that

$$j_{(\Gamma, \gamma)}(D) \in KK(C_0(T^*Z) \rtimes_{\gamma} \Gamma, C_r^*(\Gamma, \gamma)),$$

so that the above equality is meaningful. The proof is straightforward.

To show how to use this theorem, we shall combine it with the recent result of G. G. Kasparov ([19]) to compute $K_i(C_r^*(\Gamma, \gamma))$ in the following example: we let $\Gamma = \pi_1(M)$ be the fundamental group of a Riemann surface M with genus > 1 . From the exact sequence $0 \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow 0$ one gets $H^2(\Gamma, S^1) = \mathbf{R}/\mathbf{Z}$, so that there are many non trivial cocycles in this example. The geometric group $K_{\gamma}^i(\text{pt}, \Gamma)$ is easily determined: since the universal cover \tilde{M} of M (the Poincaré disc) is a final object in the category of proper Γ -manifolds, and homotopy classes of Γ -maps, it is enough to compute $K_{(\Gamma, \gamma)}^i(T^*\tilde{M})$. Since \tilde{M} has a Γ -invariant Spin^c -structure, the Thom isomorphism hence gives: $K_{\gamma}^i(\text{pt}, \Gamma) = K_{(\Gamma, \gamma)}^i(\tilde{M})$. By Proposition 1, one has $K_{(\Gamma, \gamma)}^i(\tilde{M}) = K_i(C_0(\tilde{M}) \rtimes_{\gamma} \Gamma)$ and the latter C^* -algebra is Morita equivalent to $C(M)$ (see the proof of a) in Proposition 1). Thus we get: $K_{\gamma}^0(\text{pt}, \Gamma) = \mathbf{Z}^2$, $K_{\gamma}^1(\text{pt}, \Gamma) = \mathbf{Z}^{2g}$.

THEOREM 3. *Let Γ be the fundamental group of a Riemann surface of genus > 1 , and $\gamma \in H^2(\Gamma, S^1)$, then the map $\mu: K_\gamma^*(\text{pt}, \Gamma) \rightarrow K_*(C_r^*(\Gamma, \gamma))$ is an isomorphism.*

Proof. Let $D \in KK_G(C_0(U), \mathbf{C})$ be the $G = \text{PSL}(2, \mathbf{R})$ equivariant Dirac operator on the Poincaré disc $U = G/G_c$ (cf. [19]). Identify \tilde{M} with U and Γ with a subgroup of G . Then by Proposition 1 b) and Theorem 2 it is enough to show that the restriction of D to an element of $KK_\Gamma(C_0(U), \mathbf{C})$ is an invertible element. This follows from [19] which shows that D is an invertible element of $KK_G(C_0(U), \mathbf{C})$, and from the multiplicative property of the restriction to subgroups.

We shall now show how to prove that the C^* -algebras $C_r^*(\Gamma, \gamma)$ are pairwise non-isomorphic when γ varies in $H^2(\Gamma, S^1)$. In fact we shall compute in full generality the composition $\zeta \circ \mu$ of the canonical trace ζ on $C_r^*(\Gamma, \gamma)$ (viewed as a map from K_0 to \mathbf{C}) with the above map $\mu: K_\gamma^0(\text{pt}, \Gamma) \rightarrow K_0(C_r^*(\Gamma, \gamma))$.

The computation is a generalization of the index theorem for covering spaces of Atiyah ([3]).

LEMMA 4. *Let Z be a proper Γ -manifold and E a (Γ, γ) vector bundle on Z . There exists a Γ -invariant connection ∇ on E .*

Proof. For any (Γ, γ) -vector bundle F on Z and section $\xi \in C_c^\infty(Z, F)$ let, for $g \in \Gamma$, $g\xi \in C_c^\infty(Z, F)$ be given by: $(g\xi)(x) = (\xi(xg))g^{-1} \in F_x$ for every $x \in Z$.

In this way one gets a natural γ -action of Γ on both $C_c^\infty(Z, E)$ and $C_c^\infty(Z, E \otimes T^*Z)$, and one looks for a connection

$$\nabla: C_c^\infty(Z, E) \rightarrow C_c^\infty(Z, E \otimes T^*Z)$$

such that $\nabla(g\xi) = g(\nabla\xi)$ for every ξ . Let $f \in C^\infty(Z)$, $0 \leq f \leq 1$, be such that $\sum_\Gamma f(xg) = 1$ for every $x \in Z$ and ∇_0 be a connection on E . Put $\nabla = \sum_\Gamma g^{-1}(f\nabla_0)g$. By construction ∇ is Γ -invariant, moreover each $g^{-1}\nabla_0 g$ is a connection on E thus ∇ is a connection on E . \square

Proof of Theorem 3, continued. Assuming now that Z is Γ -compact, let for a Γ -invariant connection ∇ on E , ω_∇ be the canonical differential form on Z which represents locally the Chern character $\text{ch}(E)$. By construction ω_∇ is Γ -invariant and hence determines a cohomology class in Z/Γ . One checks as usual that this class does not depend upon the choice of ∇ and

we shall denote it by $[E] \in H^*(Z/\Gamma, \mathbf{R})$. This construction easily extends to give a map ch from $K_{(\Gamma, \gamma)}^0(Z)$ to $H^*(Z/\Gamma, \mathbf{R})$ for any proper Γ -manifold Z . However, in the presence of the 2-cocycle γ the range of this map is *no longer necessarily contained* in $H^*(Z/\Gamma, \mathbf{Q})$.

To be more precise, let us make a few simplifying assumptions and compute exactly the range of this Chern character:

$$\text{ch}: K_{(\Gamma, \gamma)}^0(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R}).$$

Thus let us assume that Γ is torsion free and that the image of $\gamma \in H^2(\Gamma, S^1)$ in $H^3(\Gamma, \mathbf{Z})$ under the connecting map of the long exact sequence:

$$\dots \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow H^3(\Gamma, \mathbf{Z}) \rightarrow \dots$$

is equal to 0 (it is always a torsion element).

Let then $\rho \in H^2(\Gamma, \mathbf{R})$ be such that $e(\rho) = \gamma$ where $e: \mathbf{R} \rightarrow S^1$ is given by $e(s) = \exp(2\pi i s)$, for each $s \in \mathbf{R}$.

LEMMA 5. a) Let $\rho \in Z^2(\Gamma, \mathbf{R})$ and Z be a proper Γ -manifold, then there exists a smooth function $c \in C^\infty(Z \rtimes \Gamma)$ such that:

$$c(x, g_1) + c(xg_1, g_2) = c(x, g_1g_2) - \rho(g_1, g_2)$$

for every $x \in Z$, $g_1, g_2 \in \Gamma$.

b) If $\gamma = e(\rho)$ there exists an isomorphism $r: K_\Gamma^0(Z) \rightarrow K_{(\Gamma, \gamma)}^0(Z)$ making the following diagram commutative:

$$\begin{array}{ccc} K_\Gamma^0(Z) & \xrightarrow{r} & K_{(\Gamma, \gamma)}^0(Z) \\ \downarrow \text{ch} & & \downarrow \text{ch} \\ H^*(Z/\Gamma) & \xrightarrow{m} & H^*(Z/\Gamma), \end{array}$$

where m is multiplication by the cohomology class $\exp(\phi^*\rho)$ and where $\phi: Z/\Gamma \rightarrow B\Gamma$ is the classifying map.

Proof. a) Let $M = Z/\Gamma$, $\pi: Z \rightarrow M$ the projection. Since Z is a locally trivial Γ -principal bundle, it is easy to construct c on the open set $\pi^{-1}(U)$ for U small enough. Then one combines such c_U by a smooth partition of unity on M :

$$c(x, g) = \sum \phi_U(\pi(x)) c_U(x, g).$$

b) Let $c \in C^\infty(Z \rtimes \Gamma)$ be as in a) and let us endow the trivial line bundle on Z (with total space $Z \times \mathbf{C}$) with a structure of (Γ, γ) -bundle. We take:

$$(x, \lambda)g = (xg, e(c(x, g))\lambda).$$

(One has $((x, \lambda)g_1)g_2 = (xg_1g_2, e(c(x, g_1) + c(xg_1, g_2))\lambda) = \gamma^{-1}(g_1, g_2)(x\lambda)(g_1g_2)$.)

Let L be the (Γ, γ) -line bundle on Z thus obtained. It is obvious that tensoring by L gives an isomorphism of $V_{(\Gamma)}^0(Z)$ with $V_{(\Gamma, \gamma)}^0 Z$ and hence of $K_{\Gamma}^0(Z)$ with $K_{(\Gamma, \gamma)}^0(Z)$. \square

End of proof of Theorem 3. To conclude, it is enough to compute $\text{ch}(L)$. Let $\xi \in C^\infty(Z, L)$ be the section $\xi(x) = 1$ for every $x \in Z$. Let ∇ be a Γ -invariant connection on L , one has $\text{ch}(L) = \exp(\omega)$ where $\omega \in H^2(Z/\Gamma, \mathbf{R})$ corresponds to the Γ -invariant 2-form $\theta = \frac{1}{2\pi i} d(\nabla \xi / \xi)$ on Z . Let $\alpha = \frac{1}{2\pi i} \nabla \xi / \xi$, then α is a 1-form on Z , and let us compute for any $g \in \Gamma$ the difference $\alpha - \phi^* \alpha$ where $\phi(x) = xg$ for every $x \in Z$. Since ∇ is Γ -invariant, one has $\phi^* \alpha = \frac{1}{2\pi i} \nabla g(\xi) / g(\xi)$, and as $g(\xi)(x) = e(c(xg, g^{-1})) \xi(x)$ one gets $\phi^* \alpha - \alpha = d\psi_g$, where $\psi_g(x) = c(xg, g^{-1})$ for every $x \in Z$. One has $\psi_{g_1g_2} - g_1\psi_{g_2} - \psi_{g_1} = \rho(g_2^{-1}, g_1^{-1})$. This shows that the class of θ in $H^2(Z/\Gamma, \mathbf{R})$ is the pull back of the class of $-\rho$ in $H^2(B\Gamma, \mathbf{R})$, by the classifying map: $Z/\Gamma \rightarrow B\Gamma$. \square

Using this map $\text{ch}: K_{(\Gamma, \gamma)}^*(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R})$ we get, by the same five steps as in §6, a map

$$K_{\gamma}^*(\text{pt}, \Gamma) \xrightarrow{\text{ch}} H_*(B\Gamma, \mathbf{R}).$$

Again as in §6, let ϵ be the map from $B\Gamma$ to a point, and tr_{Γ} be the canonical trace on $C_r^*(\Gamma, \gamma)$.

THEOREM 6. *For any discrete group Γ and 2-cocycle γ the following diagram is commutative:*

$$\begin{array}{ccc} K_{\gamma}^0(\text{pt}, \Gamma) & \xrightarrow{\mu} & K_0(C_r^*(\Gamma, \gamma)) \\ \downarrow \text{ch} & & \downarrow \text{tr}_{\Gamma} \\ H_*(B\Gamma, \mathbf{R}) & \xrightarrow{\epsilon^*} & \mathbf{C}. \end{array}$$

The proof is a simple adaptation of the heat equation method to compute the Γ -index of the (Γ, γ) -Dirac operator on a Γ -manifold Z .

COROLLARY 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of \mathbf{R} , $\Delta = \text{tr}_\Gamma(K_0(C_r^*(\Gamma, \gamma)))$ contains the group:*

$$\langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle.$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma, \mathbf{R})/H^2(\Gamma, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C_r^*(\Gamma, \gamma))$ by the canonical trace $\zeta = \text{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma, \gamma)$ (for any value of γ), one gets that the corresponding C^* -algebras are isomorphic only when the Γ 's are the same (using K_1) and when the γ 's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞ -manifold, and let F be a C^∞ -foliation of V . Thus F is a C^∞ -integrable sub-vector bundle of TV . As in [33] let G be the holonomy groupoid (graph) of (V, F) . The manifold V is assumed to be Hausdorff and second countable. G , however, is a C^∞ -manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞ -paths

$$\gamma: [0, 1] \rightarrow V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0, 1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{r} \end{smallmatrix} V$.

Let Z be a possibly non-Hausdorff C^∞ -manifold. Assume given a C^∞ -map $\rho: Z \rightarrow V$, set

$$Z \circ G = \{(z, \gamma) \in Z \times G \mid \rho(z) = s(\gamma)\}.$$

A C^∞ right action of G on Z is a C^∞ -map