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where the power series converges in the domain \mathfrak{D} , and

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases} \quad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a p -adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

4. PROPERTIES OF $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ .

4.1 A SYMMETRY PROPERTY IN t

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$(18) \quad L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t),$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$(19) \quad b_n(t) = B_{n, \chi_n}(qt) - \chi_n(p) p^{n-1} B_{n, \chi_n}(p^{-1}qt),$$

and we define

$$(20) \quad c_n(t) = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} b_m(t).$$

LEMMA 4.1. For all $n \in \mathbf{Z}$, $n \geq 0$, we have

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1}.$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^n \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1}.$$

Any m such that $n - m$ is even must have the same parity as n . Thus

$$\begin{aligned} B_{n,1}(-t) &= (-1)^n \sum_{\substack{m=0 \\ n-m \text{ even}}}^n \binom{n}{m} B_{n-m,1} t^m + (-1)^{n-1} n B_{1,1} t^{n-1} \\ &= (-1)^n B_{n,1}(t) - 2(-1)^n n B_{1,1} t^{n-1}. \end{aligned}$$

From the value $B_{1,1} = -B_1 = 1/2$, the lemma then follows. \square

LEMMA 4.2. For all $n \in \mathbf{Z}$, $n \geq 0$,

$$b_n(-t) = \chi(-1)b_n(t).$$

Proof. This is obviously true for $n = 0$ since

$$b_0(t) = (1 - \chi(p)p^{-1}) B_{0,\chi},$$

and $B_{0,\chi} = 0$ except when $\chi = 1$, in which case $B_{0,1} = 1$. So we can assume that $n \geq 1$.

First consider the case of $\chi_n = 1$. This implies that $\chi = \omega^n$. By Lemma 4.1,

$$\begin{aligned} b_n(-t) &= B_{n,1}(-qt) - p^{n-1} B_{n,1}(-p^{-1}qt) \\ &= (-1)^n B_{n,1}(qt) - (-1)^n n (qt)^{n-1} \\ &\quad - p^{n-1} \left((-1)^n B_{n,1}(p^{-1}qt) - (-1)^n n (p^{-1}qt)^{n-1} \right) \\ &= (-1)^n \left(B_{n,1}(qt) - p^{n-1} B_{n,1}(p^{-1}qt) \right) \\ &= (-1)^n b_n(t). \end{aligned}$$

Since $\chi = \omega^n$ and $\omega(-1) = -1$, the lemma holds for $\chi_n = 1$.

Now suppose that $\chi_n \neq 1$. Then, from (3),

$$\begin{aligned} b_n(-t) &= B_{n,\chi_n}(-qt) - \chi_n(p)p^{n-1} B_{n,\chi_n}(-p^{-1}qt) \\ &= (-1)^n \chi_n(-1) \left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1} B_{n,\chi_n}(p^{-1}qt) \right) \\ &= (-1)^n \chi_n(-1) b_n(t). \end{aligned}$$

Note that $\chi_n = \chi\omega^{-n}$, which implies that $\chi_n(-1) = (-1)^n \chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$.

Since the lemma holds for both $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. \square

Using this result, we can prove

THEOREM 4.3. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi).$$

Proof. From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1)b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1)c_n(t).$$

From (16), whenever $n \geq -1$,

$$a_n(-t) = \chi(-1)a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi). \quad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in t will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$. Then for $n \in \mathbf{Z}$, $n \geq 1$,*

$$\lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s+n, t; \chi) = -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

Proof. Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s-1} + \sum_{m=0}^{\infty} a_m(t)(s-1)^m,$$

where $a_{-1}(t) = (1 - \chi(p)p^{-1})B_{0, \chi}$. Thus

$$\lim_{s \rightarrow 1} (s - 1)L_p(s, t; \chi) = (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

Now let $n \in \mathbf{Z}$, $n \geq 1$, and consider

$$\lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = \lim_{s \rightarrow 1} \binom{n-s}{n} L_p(s, t; \chi).$$

If $n = 1$, then we write this as

$$\lim_{s \rightarrow 1} (1 - s)L_p(s, t; \chi) = - (1 - \chi(p)p^{-1}) B_{0, \chi}.$$

If $n \geq 2$, then

$$\frac{1}{n!} \lim_{s \rightarrow 1} \prod_{i=0}^{n-2} (n - s - i) = \frac{1}{n},$$

which implies that

$$\begin{aligned} \lim_{s \rightarrow 1-n} \binom{-s}{n} L_p(s + n, t; \chi) &= \frac{1}{n!} \left(\lim_{s \rightarrow 1} \prod_{i=0}^{n-2} (n - s - i) \right) \left(\lim_{s \rightarrow 1} (1 - s)L_p(s, t; \chi) \right) \\ &= -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}. \end{aligned}$$

Therefore the lemma holds for all $n \geq 1$. \square

Now, because $L_p(s, t; 1)$ is undefined when $s = 1$, the quantity

$$\binom{-s}{n} L_p(s + n, t; 1)$$

is undefined when $s = 1 - n$, for $n \in \mathbf{Z}$, $n \geq 1$. However, Lemma 4.4 shows that this quantity exists as $s \rightarrow 1 - n$. In the following we will encounter expressions that involve $\binom{-s}{n} L_p(s + n, t; \chi)$, and because of Lemma 4.4 we shall assume the understanding that

$$\binom{-s}{n} L_p(s + n, t; \chi) \Big|_{s=1-n} = -\frac{1}{n} (1 - \chi(p)p^{-1}) B_{0, \chi}$$

for $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.5. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$(21) \quad L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m t^m L_p(s + m; \chi_m).$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $k \in \mathbf{Z}$, $k \geq 1$. Then

$$\sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-k+m; \chi_m) = -\frac{1}{k} q^k t^k (1 - \chi_k(p) p^{-1}) B_{0, \chi_k} \\ + \sum_{m=0}^{k-1} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m).$$

By evaluating the L -function, we obtain

$$\binom{k-1}{m} L_p(1-(k-m); \chi_m) = -\frac{1}{k} \binom{k}{m} (1 - \chi_k(p) p^{k-m-1}) B_{k-m, \chi_k},$$

and thus

$$\sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m) \\ = -\frac{1}{k} \sum_{m=0}^k \binom{k}{m} q^m t^m (1 - \chi_k(p) p^{k-m-1}) B_{k-m, \chi_k},$$

which implies that the sum converges for $s = 1 - k$. Breaking this into two sums

$$\sum_{m=0}^{\infty} \binom{k-1}{m} q^m t^m L_p(1-(k-m); \chi_m) \\ = -\frac{1}{k} \sum_{m=0}^k \binom{k}{m} B_{k-m, \chi_k} q^m t^m + \frac{1}{k} \chi_k(p) p^{k-1} \sum_{m=0}^k \binom{k}{m} B_{k-m, \chi_k} p^{-m} q^m t^m \\ = -\frac{1}{k} (B_{k, \chi_k}(qt) - \chi_k(p) p^{k-1} B_{k, \chi_k}(p^{-1}qt)) \\ = L_p(1-k, t; \chi).$$

Thus (21) holds for a sequence $\{1-k\}_{k=1}^{\infty}$ that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in s , of each of the functions on either side of (21) contain \mathfrak{D} , except $s \neq 1$ when $\chi = 1$.

This is obvious for the function $L_p(s, t; \chi)$. Consider the function

$$\sum_{m=0}^{\infty} \binom{-s}{m} q^m t^m L_p(s+m; \chi_m) = \sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-s}{m} q^m t^m a_{n, \chi_m} (s+m-1)^n.$$

We have seen that this sum converges for $s = 1 - k$, where $k \in \mathbf{Z}$, $k \geq 1$. Now we need to show that it converges for $s = \xi$, where $\xi \in \mathfrak{D}$, $\xi \neq 1$ if $\chi = 1$, and $\xi \neq 1 - k$ for $k \in \mathbf{Z}$, $k \geq 1$. So let ξ satisfy these restrictions,

and let $\epsilon > 0$. Note that $|\xi - 1|_p < r$, where $r = |p|_p^{1/(p-1)}|q|_p^{-1}$. Let $r_0 \in \mathbf{R}$, $0 \leq r_0 < r$, such that $|\xi - 1|_p = r_0$. Then for any $m \in \mathbf{Z}$, $m \geq 0$,

$$\begin{aligned} |\xi + m - 1|_p &\leq \max \left\{ |m|_p, |\xi - 1|_p \right\} \\ &\leq \max \{1, r_0\}, \end{aligned}$$

implying that $\xi + m \in \mathfrak{D}$, $\xi + m \neq 1$. Let $\delta \in \mathbf{R}$ such that $r^\delta = \max\{1, r_0\}$. Then $0 \leq \delta < 1$, and

$$(22) \quad |\xi + m - 1|_p \leq r^\delta.$$

Let $N_1 \in \mathbf{Z}$ such that

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $m \geq N_1$, we must also have

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For $m \in \mathbf{Z}$, $m \geq 1$, consider

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p|_p^{-1} |q|_p^m \left| \binom{-\xi}{m} (\xi + m - 1)^{-1} \right|_p.$$

Note that, by (22),

$$\begin{aligned} \left| \binom{-\xi}{m} (\xi + m - 1)^{-1} \right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^m \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\leq |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p|_p^{-1} |q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \geq |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p \leq |p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if $m \geq N_1$, then

$$\left| \binom{-\xi}{m} q^m t^m a_{-1, \chi_m} (\xi + m - 1)^{-1} \right|_p < \epsilon.$$

Now let $N_2 \in \mathbf{Z}$ such that

$$|f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)N_2/(p-1)} |q|_p^{(1-\delta)N_2} < \epsilon.$$

Then we must also have

$$|f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} < \epsilon$$

for any $m, n \in \mathbf{Z}$ such that $m \geq 0$, $n \geq 0$, and $\max\{m, n\} \geq N_2$. Let us consider

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq \left| \binom{-\xi}{m} \right|_p |q|_p^m |a_{n, \chi_m}|_p |\xi + m - 1|_p^n,$$

where $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq 0$. For all $m \geq 0$,

$$\left| \binom{-\xi}{m} \right|_p \leq |m!|_p^{-1} r^{\delta m},$$

and by utilizing this along with (17) and (22), our expression becomes

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq |m!(n+1)!|_p^{-1} |f_{\chi} p|_p^{-1} r^{\delta(m+n)} |q|_p^{m+n}.$$

Since

$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$

we obtain

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p \leq |f_{\chi} p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)}.$$

Thus if $\max\{m, n\} \geq N_2$, then

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p < \epsilon.$$

Let $N = \max\{N_1, N_2\}$, and let $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq -1$. Then for $\max\{m, n\} \geq N$, it must be true that

$$\left| \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n \right|_p < \epsilon.$$

Thus, by Proposition 2.4, the sum

$$\sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-\xi}{m} q^m t^m a_{n, \chi_m} (\xi + m - 1)^n$$

must converge. This implies that the function on the right of (21) must converge for all $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and the theorem must then hold. \square

Since we can now express $L_p(s, t; \chi)$ in terms of a power series in t , we can take a derivative of this function with respect to t .

LEMMA 4.6. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n!q^n \binom{-s}{n} L_p(s + n, t; \chi_n),$$

for $n \in \mathbf{Z}$, $n \geq 0$.

Proof. If $n = 0$, then the lemma is obviously true. So consider $n = 1$. Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s, t; \chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s + m; \chi_m).$$

Now,

$$m \binom{-s}{m} = -s \binom{-s-1}{m-1},$$

so that

$$\begin{aligned} \frac{\partial}{\partial t} L_p(s, t; \chi) &= \sum_{m=1}^{\infty} (-s) \binom{-s-1}{m-1} q^m t^{m-1} L_p(s + m; \chi_m) \\ &= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p(s + 1 + m; \chi_{1+m}) \\ &= -qs L_p(s + 1, t; \chi_1). \end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n!q^n \binom{-s}{n} L_p(s + n, t; \chi_n)$$

for some $n \in \mathbf{Z}$, $n \geq 1$. Then

$$\begin{aligned} \frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) &= \frac{\partial}{\partial t} \left(\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) \right) \\ &= n!q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s + n, t; \chi_n). \end{aligned}$$

From the case for $n = 1$, we see that

$$\begin{aligned} n!q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s + n, t; \chi_n) &= n!q^n \binom{-s}{n} (-s - n)q L_p(s + n + 1, t; \chi_{n+1}) \\ &= (n + 1)!q^{n+1} \binom{-s}{n + 1} L_p(s + n + 1, t; \chi_{n+1}). \end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) = (n + 1)!q^{n+1} \binom{-s}{n + 1} L_p(s + n + 1, t; \chi_{n+1}),$$

and the lemma must hold by induction. \square

With this result, we can derive a more general power series expansion of $L_p(s, t; \chi)$.

THEOREM 4.7. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then for $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,*

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t - \alpha)^m L_p(s + m, \alpha; \chi_m).$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s, t; \chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s, t; \chi)$ in the form

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \beta_m (t - \alpha)^m,$$

where

$$\beta_m = \frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) \Big|_{t=\alpha}.$$

From Lemma 4.6

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) = \binom{-s}{m} q^m L_p(s + m, t; \chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s + m, \alpha; \chi_m),$$

completing the proof. \square

4.3 RELATING $L_p(s, t; \chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s, t; \chi)$.

For the character χ , let $F_0 = \text{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$(23) \quad L_p(s, t + F; \chi) - L_p(s, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s}.$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then from (18),

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = -\frac{1}{n} (b_n(t + F) - b_n(t)).$$

Now, (19) implies

$$\begin{aligned} b_n(t + F) - b_n(t) &= (B_{n, \chi_n}(q(t + F)) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q(t + F))) \\ &\quad - (B_{n, \chi_n}(qt) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}qt)) \\ &= (B_{n, \chi_n}(q(t + F)) - B_{n, \chi_n}(qt)) \\ &\quad - \chi_n(p)p^{n-1} (B_{n, \chi_n}(p^{-1}q(t + F)) - B_{n, \chi_n}(p^{-1}qt)). \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} b_n(t + F) - b_n(t) &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n\chi_n(p)p^{n-1} \sum_{a=1}^{p^{-1}qF} \chi_n(a)(a + p^{-1}qt)^{n-1} \\ &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n \sum_{\substack{a=1 \\ p|a}}^{qF} \chi_n(a)(a + qt)^{n-1}. \end{aligned}$$

Therefore,

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a + qt)^{n-1}.$$

Now, $\chi_n = \chi_1\omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a + qt)^{n-1} &= \chi_1(a)\omega^{-(n-1)}(a)(a + qt)^{n-1} \\ &= \chi_1(a)\langle a + qt \rangle^{n-1}. \end{aligned}$$

Thus

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a + qt \rangle^{n-1},$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable s , of the functions on the left of (23) contain \mathfrak{D} , except $s \neq 1$ when $\chi = 1$. Consider now the function

$$- \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-1} \langle a + qt \rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a + qt \rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a, p) = 1$, we need only show that each such function is analytic on \mathfrak{D} , and the proof will be complete.

The quantity $\langle a + qt \rangle^{1-s}$ can be written as

$$\langle a + qt \rangle^{1-s} = \exp((1-s) \log \langle a + qt \rangle),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a + qt \rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m.$$

Since $\langle a + qt \rangle \equiv 1 \pmod{qo}$ for $a \in \mathbf{Z}$, $(a, p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log \langle a + qt \rangle \equiv 0 \pmod{qo}$ for such a and t . Thus

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \leq \left| \frac{1}{m!} q^m (s-1)^m \right|_p$$

for all m . By (8) we can write

$$\begin{aligned} \left| \frac{1}{m!} q^m (s-1)^m \right|_p &\leq \left| p^{-m/(p-1)} q^m (s-1)^m \right|_p \\ &= \left| p^{-1/(p-1)} q (s-1) \right|_p^m. \end{aligned}$$

Thus if

$$\left| p^{-1/(p-1)} q (s-1) \right|_p < 1,$$

then

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \rightarrow 0$$

as $m \rightarrow \infty$. So whenever $|s-1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain \mathfrak{D} , except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. \square

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \langle a \rangle^{-s}.$$

Proof. This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character χ . \square

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let Δ_c denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \text{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n} \left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}(p^{-1}qt) \right).$$

This is the polynomial structure that we utilized with respect to generalizing the p -adic L -functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either χ or p .

THEOREM 4.10. *Let n , c , and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n .*

Proof. Since Δ_c is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \Delta_c^k \langle a \rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \langle a \rangle^{-1} \Delta_c^k \langle a \rangle^n.$$

Note that

$$(24) \quad \Delta_c^k \langle a \rangle^n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} \langle a \rangle^{n+mc} = \langle a \rangle^n (\langle a \rangle^c - 1)^k.$$

Now, $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a \rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k \langle a \rangle^n \equiv 0 \pmod{q^k \mathbf{Z}_p}.$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Also, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$,

$$(25) \quad q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^k$$

implies that the value of $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n .

Let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i) \rightarrow \beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i \rightarrow \infty} (\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0)) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0).$$

The left side of this equality is 0 modulo $q^k \mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Furthermore, for n' a positive integer,

$$\begin{aligned} & \lim_{i \rightarrow \infty} ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))) \\ &= ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))). \end{aligned}$$

Since $\tau_i \in pq^{-1}F_0\mathbf{Z}$ for each i , the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n . \square

THEOREM 4.11. *Let $n, c, k,$ and k' be positive integers with $k \equiv k' \pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then*

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

Proof. Let k and k' be positive integers such that $k \equiv k' \pmod{p-1}$. Without loss of generality, we can assume that $k \geq k'$. From (25),

$$\begin{aligned} & (q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)) \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^k + \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^{k'} \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^{k'} \left(\left(\frac{\langle a \rangle^c - 1}{q}\right)^{k-k'} - 1 \right), \end{aligned}$$

where F is a positive multiple of $pq^{-1}F_0$. If a is such that

$$\langle a \rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a \rangle^c - 1}{q}\right)^{k-k'} - 1 \equiv 0 \pmod{p\mathbf{Z}_p},$$

since $k - k' \equiv 0 \pmod{p-1}$. Thus

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^\infty$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. Consider

$$\begin{aligned} \lim_{i \rightarrow \infty} ((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))) \\ = (q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)). \end{aligned}$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. \square

THEOREM 4.12. Let n , c , and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n .

Proof. We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k} L_p(1-n, F; \chi) = \binom{q^{-1}\Delta_c}{k} L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \binom{q^{-1}\Delta_c}{k} \langle a \rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{-1} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n.$$

Utilizing (15), we can write

$$\begin{aligned} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \Delta_c^m \langle a \rangle^n \\ &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \langle a \rangle^n (\langle a \rangle^c - 1)^m, \end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k} \langle a \rangle^n = \langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k}.$$

Since $q^{-1}(\langle a \rangle^c - 1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a, p) = 1$, we see that

$$\langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of n .

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^\infty$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. We are working with polynomials, so that

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) \\ = \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0), \end{aligned}$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right) \\ = \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right). \end{aligned}$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n . \square

4.4 GENERALIZED BERNOULLI POWER SERIES

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n} , where $n \in \mathbf{Z}$, $n \geq 1$, in the field \mathbf{Q}_p , given by

$$(26) \quad B_{-n} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n},$$

where the limit is taken in a p -adic sense. Note that $\phi(p^k) \rightarrow 0$ in \mathbf{Z}_p as $k \rightarrow \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k \rightarrow \infty} \left(1 - p^{\phi(p^k)-n-1} \right) B_{\phi(p^k)-n} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \omega^{-n}) \\ &= nL_p(n + 1; \omega^{-n}). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p,$$

where each prime r is taken to be a rational prime.

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) | n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

Proof. By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) | \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) | n$ if and only if $(p-1) | (\phi(p^k) - n)$. Therefore, if k is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p,$$

and the result follows from (26). \square

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field \mathbf{C}_p according to

$$(27) \quad B_{-n,\chi} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n,\chi},$$

where the limit is once again taken in a p -adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m,\chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters χ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n,\chi} &= \lim_{k \rightarrow \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1} \right) B_{\phi(p^k)-n,\chi_{\phi(p^k)}} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \chi_n) \\ &= nL_p(n+1; \chi_n), \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n,1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $B_{-n,1} = B_{-n}$ for all such n .

If $k \geq 2$, then $\phi(p^k)$ is even. Thus n and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even.} \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod{2}$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod{2}$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let χ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

Proof. Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k \rightarrow \infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the p -adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the p -adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. \square

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k \rightarrow \infty} (B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p)p^{\phi(p^k)-n-1} B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)) \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n)L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= nL_p(n + 1, t; \chi_n). \end{aligned}$$

Since $L_p(n + 1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such t . Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p(n + m + 1; \chi_{n+m}) \\ &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n + m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \leq \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$(28) \quad B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \geq 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}(mf_\chi) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

Proof. By definition, since $|mf_\chi|_p \leq |q|_p$,

$$\begin{aligned} B_{-n,\chi}(mf_\chi) - B_{-n,\chi}(0) &= \lim_{k \rightarrow \infty} (B_{\phi(p^k)-n,\chi}(mf_\chi) - B_{\phi(p^k)-n,\chi}(0)) \\ &= \lim_{k \rightarrow \infty} (\phi(p^k) - n) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k - 1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a, p) = 1$. These imply that

$$\lim_{k \rightarrow \infty} (\phi(p^k) - n) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. \square

THEOREM 4.16. Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all χ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t).$$

Proof. Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod{2}$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n + \delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t), \end{aligned}$$

and the proof is complete. \square

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