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Autor: Fox, Glenn J.

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and thus

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

The uniqueness of $L_p(s, \tau; \chi)$ follows from Lemma 2.5. □

At this point we have not completed our goal of showing that the p-adic function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. In order to prove this, we will need to study the coefficients, $a_n(\tau)$, of the power series expansion of $L_p(s, \tau; \chi)$ for each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. From the results of this we will show that the function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for any sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to τ, the values $L_p(1 - n, \tau_i; \chi)$ converge to $L_p(1 - n, \tau; \chi)$ for each $n \in \mathbf{Z}$, $n \geq 1$.

3.2 $L_p(s, \tau; \chi)$ FOR $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$

Our previous work has been for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. To extend this result to all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we need to find a way to express $a_n(\tau)$ so that it can be defined for these values of τ.

For $k \in \mathbf{Z}$, $k \geq 0$, the Stirling numbers of the first kind, $s(n, k)$, are defined by the generating function

$$\sum_{n=0}^{\infty} s(n, k) \frac{t^n}{n!} = \frac{1}{k!} (\log(1 + t))^k . \tag{14}$$

Since the power series expansion of $\log(1 + t)$ lacks a constant term, we must have $s(n, k) = 0$ whenever $0 \leq n < k$. We also have $s(n, n) = 1$ for all $n \geq 0$. The $s(n, k)$ are integers, where $n, k \in \mathbf{Z}$, $n \geq 0$, $k \geq 0$, and they satisfy the relation

$$\binom{x}{n} = \frac{1}{n!} \sum_{k=0}^{n} s(n, k) x^k . \tag{15}$$

For additional information on Stirling numbers of the first kind we refer the reader to [6], pp. 214–217.

LEMMA 3.6. *Let* $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. *For* $n \in \mathbf{Z}$, $n \geq -1$,

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n + 1) c_m(\tau) .$$

Proof. From Corollary 2.8 we can write

$$A_\chi(s,\tau)=\sum_{m=0}^{\infty}\binom{s}{m}c_m(\tau),$$

where $s\in\mathbf{C}_p$ such that $|s|_p<|p|_p^{1/(p-1)}|q|_p^{-1}$. Now, expanding the quantity $\binom{s}{m}$ according to (15) yields

$$A_\chi(s,\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}s(m,n)c_m(\tau)s^n,$$

where $s(m,n)\in\mathbf{Z}$ is a Stirling number of the first kind. At this point we wish to switch the order of summation in this expression, but before doing so we must show that the terms in the summation converge to 0 at a sufficient rate.

Let $\epsilon>0$ and let $\xi\in\mathbf{C}_p$ such that $|\xi|_p<|p|_p^{1/(p-1)}|q|_p^{-1}$. Then there exists $\delta\in\mathbf{R}$, $0\le\delta<1$, such that

$$|\xi|_p=\delta\cdot|p|_p^{1/(p-1)}|q|_p^{-1}.$$

Let $N,M\in\mathbf{Z}$, $N>0$, $M>0$, such that if $n\ge N$ then $|pqf_\chi|_p^{-1}\delta^n<\epsilon$, and if $m\ge M$ then $|pqf_\chi|_p^{-1}|p|_p^{-m/(p-1)}|q|_p^m<\epsilon$ (such an M exists since $0\le|p|_p^{-1/(p-1)}|q|_p<1$).

Let $m,n\in\mathbf{Z}$, $m\ge0$, $n\ge0$. If $n>m$, then $s(m,n)=0$, and so

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p=0.$$

Thus we can assume that $m=\max\{m,n\}$. Consider

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p\le|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n.$$

Utilizing Proposition 3.3 and the fact that $v_p(m!)\le m/(p-1)$, we can write

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n\le|pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n}\delta^n.$$

Suppose that $m\ge M+N$. If $m-n<M$, then

$$M+N\le m<M+n,$$

so that $n>N$. Thus

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n\le|pqf_\chi|_p^{-1}\delta^n<\epsilon.$$

If $m-n\ge M$, then

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n \leq |pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n} < \epsilon .$$

Either case implies that

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p < \epsilon .$$

Therefore, whenever $\max\{m,n\} \geq M+N$, this bound must hold, implying that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n,$$

by Proposition 2.4.

Writing

$$A_\chi(s,\tau) = \sum_{n=0}^{\infty}s^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau),$$

we have from (13),

$$\begin{aligned} L_p(s,\tau;\chi) &= \frac{1}{s-1}\sum_{n=0}^{\infty}(1-s)^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau) \\ &= \sum_{n=-1}^{\infty}(-1)^{n+1}(s-1)^n\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau), \end{aligned}$$

which implies the lemma, since we must have convergence for the inner sum. □

Since we have only derived $L_p(s,\tau;\chi)$ for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, we cannot say that $a_n(\tau)$ is defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$, let us define

$$a_n(\tau) = (-1)^{n+1}\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau), \tag{16}$$

for these values of τ. Note that in the proof of Lemma 3.6, the only influence generated by the value of τ is in the bound of the value of $|c_m(\tau)|_p$, which was determined in Proposition 3.3. However, this proposition holds for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Thus this sum converges and $a_n(\tau)$ is well-defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

THEOREM 3.7. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Then for $n \in \mathbf{Z}$, $n \geq -1$,*

$$\lim_{i\to\infty} a_n(\tau_i) = a_n(\tau).$$

Proof. By definition, for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for $n \in \mathbf{Z}$, $n \geq -1$, we have the expansion

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau),$$

and as we have seen, regardless of the value of τ,

$$\left| \frac{1}{m!} s(m, n+1) c_m(\tau) \right|_p \leq |pqf_\chi|_p^{-1} |p|_p^{-m/(p-1)} |q|_p^m \to 0$$

as $m \to \infty$. Therefore given $\epsilon > 0$ there must exist some $m_0 \in \mathbf{Z}$, $m_0 \geq n+1$, such that

$$\left| \sum_{m=m_0+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau) \right|_p < \epsilon.$$

Thus for any sequence $\{\tau_i\}_{i=1}^{\infty}$ in $\mathbf{Q}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$,

$$|a_n(\tau) - a_n(\tau_i)|_p \leq \max_{n+1 \leq m \leq m_0} \left\{ \epsilon, \left| \frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right) \right|_p \right\}.$$

Since $\tau_i \to \tau$ and $c_m(\tau)$ is a polynomial in τ, we see that

$$\left| \frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right) \right|_p < \epsilon$$

for all m with $n+1 \leq m \leq m_0$ when i is sufficiently large, which implies that

$$|a_n(\tau) - a_n(\tau_i)|_p < \epsilon$$

for such i. Therefore the theorem must hold. □

The purpose of the following three lemmas is to build an upper bound for the value of $|a_n(\tau)|_p$. After doing so we can define $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

LEMMA 3.8. *Let p be prime. If $i, n \in \mathbf{Z}$ with $1 \leq i \leq n$, then*

$$\left|\binom{n}{i} p^i\right|_p \leq |np|_p .$$

Proof. For $i \in \mathbf{Z}$ such that $1 \leq i \leq n$, (8) implies that $v_p(i!) \leq i-1$, or equivalently, $|i!|_p \geq |p|_p^{i-1}$. Therefore by combining this with

$$\left|\binom{n}{i} p^i\right|_p = \left|\frac{n(n-1)\cdots(n-i+1)}{i!}\right|_p |p|_p^i \leq |i!|_p^{-1} |n|_p |p|_p^i ,$$

the result will follow. □

LEMMA 3.9. *Let p be prime. Then for $m, n \in \mathbf{Z}$, $m > n \geq 0$,*

$$\left|\frac{n!}{m!} s(m,n) q^m\right|_p \leq |np|_p |q|_p^n .$$

Proof. From (14), the generating function for the $s(m,n)$, we obtain

$$\sum_{m=0}^{\infty} \frac{n!}{m!} s(m,n) q^m t^m = (\log(1+qt))^n .$$

Thus we wish to evaluate the power of p that divides the coefficient of t^m in the expansion of $(\log(1+qt))^n$. The power series expansion of the logarithm function (10) yields

$$(\log(1+qt))^n = \left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} q^i t^i\right)^n ,$$

and by factoring qt out of the sum,

$$(\log(1+qt))^n = q^n t^n \left(1 + pt \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{i} p^{-1} q^{i-1} t^{i-2}\right)^n .$$

For $i \geq 2$, we see that $p^{-1} q^{i-1}/i \in \mathbf{Z}_p$. Therefore

$$(\log(1+qt))^n = q^n t^n (1 + ptf(t))^n,$$

where $f(t) \in \mathbf{Z}_p[[t]]$. Now, this can be written

$$(\log(1+qt))^n = q^n t^n + q^n t^n \sum_{i=1}^{n} \binom{n}{i} p^i t^i f(t)^i,$$

and from Lemma 3.8, the p-adic absolute value of the coefficients of the terms in the sum on the right must be bounded above by $|np|_p |q|_p^n$. Thus, for $m > n$, the coefficient of t^m must also be bounded above by this quantity, implying the result. □

LEMMA 3.10. *Let* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. *Then for* $n \in \mathbf{Z}$, $n \geq 0$,

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}.$$

Proof. From (16), we see that for $n \in \mathbf{Z}$, $n \geq 0$,

$$f_\chi n! a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{n!}{m!} f_\chi s(m, n+1) c_m(\tau).$$

Proposition 3.3 implies that

$$f_\chi c_m(\tau) \equiv 0 \pmod{p^{-1} q^{m-1} \mathfrak{o}}.$$

By Lemma 3.9, when $m \geq n+2$,

$$\frac{n!}{m!} s(m, n+1) \equiv 0 \pmod{p q^{n-m+1} \mathfrak{o}}.$$

Thus

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}. \qquad \square$$

We are nearing our goal of defining $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. The final step before doing so is proving the following lemma on the convergence of a specific infinite sum.

LEMMA 3.11. *Let* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. *Then the sum*

$$\sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

converges for all $s \in \mathfrak{D}$.

Proof. Let $\xi \in \mathfrak{D}$. Then $|\xi - 1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$. Thus there must be some $\delta \in \mathbf{R}$, $0 \leq \delta < 1$, such that

$$|\xi - 1|_p = \delta \cdot |p|_p^{1/(p-1)} |q|_p^{-1}.$$

Let $n \in \mathbf{Z}$, $n \geq 0$. From Lemma 3.10

$$f_\chi (n+1)! a_n(\tau) \equiv (-1)^{n+1} f_\chi c_{n+1}(\tau) \pmod{(n+1) q^n \mathfrak{o}},$$

and from Proposition 3.3,

$$|f_\chi c_{n+1}(\tau)|_p \leq |p|_p^{-1} |q|_p^n.$$

Therefore

$$|f_\chi (n+1)! a_n(\tau)|_p \leq |p|_p^{-1} |q|_p^n,$$

which implies that

$$|a_n(\tau)|_p \leq |f_\chi (n+1)! p|_p^{-1} |q|_p^n .$$

Thus

$$|a_n(\tau)(\xi-1)^n|_p \leq |f_\chi (n+1)! p|_p^{-1} |p|_p^{n/(p-1)} \delta^n .$$

Now,

$$v_p((n+1)!) \leq \frac{n}{p-1},$$

so that

$$|a_n(\tau)(\xi-1)^n|_p \leq |f_\chi p|_p^{-1} \delta^n .$$

Since $0 \leq \delta < 1$, we see that $|a_n(\tau)(\xi-1)^n|_p \to 0$ as $n \to \infty$. Thus the sum

$$\sum_{n=0}^{\infty} a_n(\tau)(\xi-1)^n$$

must converge. ☐

Note that from this proof we have obtained the bound

$$|a_n(\tau)|_p \leq |f_\chi (n+1)! p|_p^{-1} |q|_p^n , \tag{17}$$

for each $n \in \mathbf{Z}$, $n \geq -1$, and for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

Now let us define

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$. This definition is consistent with what we already have for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. We will now show that, for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, this function satisfies

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. To do this, we prove the following:

LEMMA 3.12. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \le 1$, such that $\tau_i \to \tau$. Then for each $n \in \mathbf{Z}$, $n \ge 1$,*

$$\lim_{i\to\infty} L_p(1-n, \tau_i; \chi) = L_p(1-n, \tau; \chi).$$

Proof. We can write

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{m=0}^{\infty} a_m(\tau)(s-1)^m,$$

where the power series converges for each $s \in \mathfrak{D}$.

Let $\epsilon > 0$, and let $n \in \mathbf{Z}$, $n \ge 1$. Then we must have $1-n \in \mathfrak{D}$, and thus the power series converges for $s = 1-n$. Also, by (17)

$$|a_m(\tau)(-n)^m|_p \le |f_\chi(m+1)!p|_p^{-1}|nq|_p^m \to 0$$

independently of τ as $m \to \infty$. Therefore, for $m_0 \in \mathbf{Z}$ sufficiently large,

$$\left|\sum_{m=m_0}^{\infty} a_m(\tau)(-n)^m\right|_p < \epsilon.$$

For $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, let $\{\tau_i\}_{i=1}^{\infty}$ be in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \le 1$, such that $\tau_i \to \tau$. Consider

$$|L_p(1-n,\tau;\chi) - L_p(1-n,\tau_i;\chi)|_p \le \max_{0\le m<m_0}\left\{\epsilon, |(a_m(\tau)-a_m(\tau_i))(-n)^m|_p\right\}.$$

Since $a_m(\tau_i) \to a_m(\tau)$ as $\tau_i \to \tau$, we have

$$|L_p(1-n,\tau;\chi) - L_p(1-n,\tau_i;\chi)|_p < \epsilon$$

for i sufficiently large. Thus the lemma must hold. ☐

At this point we have finally proven

THEOREM 3.13. *For each $\tau \in \mathbf{C}_p$, with $|\tau|_p \le 1$, there exists a unique p-adic, meromorphic function $L_p(s, \tau; \chi)$ that satisfies*

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \ge 1$. Furthermore, this function can be expressed in the form

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *and*

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases} \qquad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a p-adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

4. PROPERTIES OF $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ.

4.1 A SYMMETRY PROPERTY IN t

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t) , \tag{18}$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right) , \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t) . \tag{20}$$

LEMMA 4.1. *For all* $n \in \mathbf{Z}$, $n \geq 0$, *we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1} .$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1} .$$