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3. THE p -ADIC L -FUNCTION $L_p(s, t; \chi)$

In the following, we apply Theorem 2.7 to the sequence $\{b_n(\tau)\}_{n=0}^\infty$, where $b_n(\tau) = B_{n, \chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q\tau)$, for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, to show that there exists a power series $A_\chi(s, \tau) \in K_\tau[[s]]$, $K_\tau = \mathbf{Q}_p(\chi, \tau)$, which converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. From this we can prove the existence of a p -adic function, $L_p(s, \tau; \chi)$, that interpolates the values $L_p(1 - n, \tau; \chi) = -\frac{1}{n}b_n(\tau)$ for $n \in \mathbf{Z}$, $n \geq 1$, and converges in $\{s \in \mathbf{C}_p : |s - 1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \neq 1$ if $\chi = 1$. After this we will show that there exists $L_p(s, \tau; \chi)$ for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, satisfying

$$L_p(1 - n, \tau; \chi) = -\frac{1}{n}b_n(\tau),$$

and converging in the domain above.

3.1 $L_p(s, \tau; \chi)$ FOR $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$

Let p be prime, and let χ be a Dirichlet character with conductor f_χ . Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, and let $K_\tau = \mathbf{Q}_p(\chi, \tau)$, the field generated over \mathbf{Q}_p by adjoining τ and the values $\chi(a)$, $a \in \mathbf{Z}$. Since τ and each of the $\chi(a)$ are in $\overline{\mathbf{Q}}_p$, we see that K_τ is a finite extension of \mathbf{Q}_p in $\overline{\mathbf{Q}}_p$. For each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, we shall derive our L -function $L_p(s, \tau; \chi)$ in a manner similar to that given for the derivation of $L_p(s; \chi)$ found in Chapter 3 of [13].

For $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, define the sequences $\{b_n(\tau)\}_{n=0}^\infty$ and $\{c_n(\tau)\}_{n=0}^\infty$ in K_τ according to

$$b_n(\tau) = B_{n, \chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q\tau),$$

and

$$c_n(\tau) = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} b_m(\tau).$$

In order to derive our L -function $L_p(s, \tau; \chi)$, we will prove a particular bound on the magnitude of $c_n(\tau)$, but to do so, we shall need the following:

LEMMA 3.1. *Let $m, r \in \mathbf{Z}$, with $m \geq 0$ and $r \geq 1$. Then*

$$\sum_{a=0}^{p^r-1} a^m \equiv 0 \pmod{p^{r-1}},$$

where we take $0^0 = 1$ in the case of $a = 0$ and $m = 0$.

Proof. This is obvious for $m = 0$, so assume that $m \geq 1$. We shall prove this result for the remaining values of m by induction on r .

Since any sum of elements of \mathbf{Z} must also be in \mathbf{Z} , the lemma is true for $r = 1$. Now assume that the lemma holds for some $r \in \mathbf{Z}$, $r \geq 1$. By rewriting the sum

$$\sum_{a=0}^{p^{r+1}-1} a^m = \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} (u + p^r v)^m,$$

and reducing this modulo p^r , we obtain

$$\begin{aligned} \sum_{a=0}^{p^{r+1}-1} a^m &\equiv \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} u^m \pmod{p^r} \\ &\equiv p \sum_{u=0}^{p^r-1} u^m \pmod{p^r}. \end{aligned}$$

By our induction hypothesis we must then have

$$\sum_{a=0}^{p^{r+1}-1} a^m \equiv 0 \pmod{p^r},$$

and the lemma follows. \square

LEMMA 3.2. Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 0$. For all $h \in \mathbf{Z}$, $h \geq 1$,

$$\frac{1}{q^h f_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) (\langle a + q\tau \rangle - 1)^n \equiv 0 \pmod{f_\chi^{-1} p^{-1} q^{n-1} \mathfrak{o}}.$$

Proof. This is obvious for $n = 0$ since writing

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \sum_{a=1}^{q^h f_\chi} \chi(a) - \sum_{a=1}^{p^{-1} q^h f_\chi} \chi(pa)$$

allows us to derive

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \begin{cases} p^{-1} q^h (p-1), & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

So let us assume that $n \geq 1$.

Let $h = 1$. Then $\langle a + q\tau \rangle \equiv 1 \pmod{q\mathfrak{o}}$ for all $a \in \mathbf{Z}$ such that $(a, p) = 1$ implies that

$$(\langle a + q\tau \rangle - 1)^n \equiv 0 \pmod{q^n\mathfrak{o}},$$

and the lemma holds for this case.

Now assume that $h \geq 1$. We can rewrite our sum as follows:

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) (\langle a + q\tau \rangle - 1)^n = \sum_{v=0}^{q^{h-1}-1} \sum_{\substack{u=1 \\ (u+vqf_\chi, p)=1}}^{qf_\chi} \chi(u + vqf_\chi) (\langle u + vqf_\chi + q\tau \rangle - 1)^n.$$

Since $|\tau|_p \leq 1$, we can write

$$\begin{aligned} \langle u + vqf_\chi + q\tau \rangle &= (u + vqf_\chi + q\tau) \omega^{-1} (u + vqf_\chi + q\tau) \\ &= (u + q\tau) \omega^{-1} (u + q\tau) + vqf_\chi \omega^{-1} (u + q\tau) \\ &= \langle u + q\tau \rangle + vqf_\chi \omega^{-1} (u). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) (\langle a + q\tau \rangle - 1)^n &= \sum_{\substack{u=1 \\ (u,p)=1}}^{qf_\chi} \chi(u) \sum_{v=0}^{q^{h-1}-1} (\langle u + q\tau \rangle - 1 + vqf_\chi \omega^{-1} (u))^n. \end{aligned}$$

By expanding, the inner sum on the right can be written

$$\begin{aligned} \sum_{v=0}^{q^{h-1}-1} (\langle u + q\tau \rangle - 1 + vqf_\chi \omega^{-1} (u))^n &= \sum_{k=0}^n \binom{n}{k} (\langle u + q\tau \rangle - 1)^{n-k} q^k f_\chi^k \omega^{-k} (u) \sum_{v=0}^{q^{h-1}-1} v^k. \end{aligned}$$

Since $(u, p) = 1$, we obtain the equivalence

$$q^k (\langle u + q\tau \rangle - 1)^{n-k} \equiv 0 \pmod{q^n\mathfrak{o}}$$

for each k , $0 \leq k \leq n$. Furthermore, by Lemma 3.1

$$\sum_{v=0}^{q^{h-1}-1} v^k \equiv 0 \pmod{p^{-1}q^{h-1}}$$

for each such k . Therefore

$$\sum_{v=0}^{q^{h-1}-1} (\langle u + q\tau \rangle - 1 + vqf_\chi \omega^{-1}(u))^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}}.$$

This implies that

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) (\langle a + q\tau \rangle - 1)^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}},$$

yielding the result. \square

We now derive our bound on the magnitude of $c_n(\tau)$.

PROPOSITION 3.3. *For all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for $n \in \mathbf{Z}$, $n \geq 0$, we have $|c_n(\tau)|_p \leq |pqf_\chi|_p^{-1} |q|_p^n$.*

Proof. This follows in a manner similar to that given for the proof of the bound $|c_n(0)|_p \leq |q^2 f_\chi|_p^{-1} |q|_p^n$ found in [13] (Lemma 4 of Chapter 3). However, in this case we use Lemma 2.3 and the properties of χ and ω to derive

$$b_n(\tau) = \lim_{h \rightarrow \infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) \langle a + q\tau \rangle^n$$

for each $n \geq 0$, and thus

$$c_n(\tau) = \lim_{h \rightarrow \infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) (\langle a + q\tau \rangle - 1)^n$$

for each such n . From Lemma 3.2 we obtain

$$c_n(\tau) \equiv 0 \pmod{f_\chi^{-1} p^{-1} q^{n-1} \mathfrak{o}},$$

and thus the result. \square

For our immediate concern we only need this proposition to hold for all $\tau \in \overline{\mathbf{Q}}_p$ such that $|\tau|_p \leq 1$. However, later on we shall need it in the form in which we have it.

We are now ready to begin the construction of our L -function.

THEOREM 3.4. *For each $\tau \in \overline{\mathbf{Q}}_p$, with $|\tau|_p \leq 1$, there exists a power series $A_\chi(s, \tau)$ in $K_\tau[[s]]$ such that the power series converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, and for each $n \in \mathbf{Z}$, $n \geq 0$, $A_\chi(n, \tau)$ satisfies*

$$A_\chi(n, \tau) = B_{n, \chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q\tau) .$$

Proof. By Proposition 3.3, $|c_n(\tau)|_p \leq C|q|_p^n$ for all $n \geq 0$, where $C = |pqf_\chi|_p^{-1}$. Therefore we can apply Theorem 2.7 to the sequences $\{b_n(\tau)\}_{n=0}^\infty$ and $\{c_n(\tau)\}_{n=0}^\infty$ in $K_\tau = \mathbf{Q}_p(\chi, \tau)$, and for $\rho = |q|_p < |p|_p^{1/(p-1)}$, yielding this result. \square

Let us denote $\mathfrak{D} = \{s \in \mathbf{C}_p : |s - 1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$.

THEOREM 3.5. *For each $\tau \in \overline{\mathbf{Q}}_p$, with $|\tau|_p \leq 1$, there exists a unique p -adic, meromorphic function $L_p(s, \tau; \chi)$ that can be expressed in the form*

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s - 1} + \sum_{n=0}^\infty a_n(\tau)(s - 1)^n,$$

where the power series converges in the domain \mathfrak{D} , having coefficients $a_n(\tau) \in \mathbf{Q}_p(\chi, \tau)$, with

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

Furthermore, for each $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1 - n, \tau; \chi) = -\frac{1}{n} (B_{n, \chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q\tau)) .$$

Proof. Let

$$(13) \quad L_p(s, \tau; \chi) = \frac{1}{s - 1} A_\chi(1 - s, \tau)$$

with the $A_\chi(s, \tau)$ as in Theorem 3.4. Then from the properties of $A_\chi(s, \tau)$, the power series must converge in the given domain, and for $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1 - n, \tau; \chi) = -\frac{1}{n} A_\chi(n, \tau) = -\frac{1}{n} (B_{n, \chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q\tau)) .$$

Note that

$$\begin{aligned} a_{-1}(\tau) &= A_\chi(0, \tau) = B_{0, \chi}(q\tau) - \chi(p)p^{-1}B_{0, \chi}(p^{-1}q\tau) \\ &= (1 - \chi(p)p^{-1})B_{0, \chi}, \end{aligned}$$

and thus

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

The uniqueness of $L_p(s, \tau; \chi)$ follows from Lemma 2.5. \square

At this point we have not completed our goal of showing that the p -adic function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. In order to prove this, we will need to study the coefficients, $a_n(\tau)$, of the power series expansion of $L_p(s, \tau; \chi)$ for each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. From the results of this we will show that the function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for any sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to τ , the values $L_p(1-n, \tau_i; \chi)$ converge to $L_p(1-n, \tau; \chi)$ for each $n \in \mathbf{Z}$, $n \geq 1$.

3.2 $L_p(s, \tau; \chi)$ FOR $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$

Our previous work has been for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. To extend this result to all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we need to find a way to express $a_n(\tau)$ so that it can be defined for these values of τ .

For $k \in \mathbf{Z}$, $k \geq 0$, the Stirling numbers of the first kind, $s(n, k)$, are defined by the generating function

$$(14) \quad \sum_{n=0}^{\infty} s(n, k) \frac{t^n}{n!} = \frac{1}{k!} (\log(1+t))^k.$$

Since the power series expansion of $\log(1+t)$ lacks a constant term, we must have $s(n, k) = 0$ whenever $0 \leq n < k$. We also have $s(n, n) = 1$ for all $n \geq 0$. The $s(n, k)$ are integers, where $n, k \in \mathbf{Z}$, $n \geq 0$, $k \geq 0$, and they satisfy the relation

$$(15) \quad \binom{x}{n} = \frac{1}{n!} \sum_{k=0}^n s(n, k) x^k.$$

For additional information on Stirling numbers of the first kind we refer the reader to [6], pp. 214–217.

LEMMA 3.6. *Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$,*

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau).$$

Proof. From Corollary 2.8 we can write