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3. THE p-ADIC L-FUNCTION $L_p(s,t;\chi)$

In the following, we apply Theorem 2.7 to the sequence $\{b_n(\tau)\}_{n=0}^{\infty}$, where $b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)$, for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, to show that there exists a power series $A_\chi(s,\tau) \in K_\tau[[s]]$, $K_\tau = \mathbf{Q}_p(\chi,\tau)$, which converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. From this we can prove the existence of a p-adic function, $L_p(s,\tau;\chi)$, that interpolates the values $L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau)$ for $n \in \mathbf{Z}$, $n \ge 1$, and converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \ne 1$ if $\chi = 1$. After this we will show that there exists $L_p(s,\tau;\chi)$ for each $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, satisfying

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau),$$

and converging in the domain above.

3.1 $L_p(s,\tau;\chi)$ FOR $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$

Let p be prime, and let χ be a Dirichlet character with conductor f_χ. Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, and let $K_\tau = \mathbf{Q}_p(\chi,\tau)$, the field generated over $\mathbf{Q}_p$ by adjoining τ and the values $\chi(a)$, $a \in \mathbf{Z}$. Since τ and each of the $\chi(a)$ are in $\overline{\mathbf{Q}}_p$, we see that K_τ is a finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$. For each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, we shall derive our L-function $L_p(s,\tau;\chi)$ in a manner similar to that given for the derivation of $L_p(s;\chi)$ found in Chapter 3 of [13].

For $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, define the sequences $\{b_n(\tau)\}_{n=0}^{\infty}$ and $\{c_n(\tau)\}_{n=0}^{\infty}$ in K_τ according to

$$b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right),$$

and

$$c_n(\tau) = \sum_{m=0}^{n} \binom{n}{m}(-1)^{n-m}b_m(\tau).$$

In order to derive our L-function $L_p(s,\tau;\chi)$, we will prove a particular bound on the magnitude of $c_n(\tau)$, but to do so, we shall need the following:

LEMMA 3.1. *Let $m, r \in \mathbf{Z}$, with $m \ge 0$ and $r \ge 1$. Then*

$$\sum_{a=0}^{p^r-1} a^m \equiv 0 \pmod{p^{r-1}},$$

where we take $0^0 = 1$ in the case of $a = 0$ and $m = 0$.

Proof. This is obvious for $m = 0$, so assume that $m \geq 1$. We shall prove this result for the remaining values of m by induction on r.

Since any sum of elements of $\mathbf{Z}$ must also be in $\mathbf{Z}$, the lemma is true for $r = 1$. Now assume that the lemma holds for some $r \in \mathbf{Z}$, $r \geq 1$. By rewriting the sum

$$\sum_{a=0}^{p^{r+1}-1} a^m = \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} \left(u + p^r v\right)^m,$$

and reducing this modulo p^r, we obtain

$$\begin{aligned} \sum_{a=0}^{p^{r+1}-1} a^m &\equiv \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} u^m \pmod{p^r} \\ &\equiv p \sum_{u=0}^{p^r-1} u^m \pmod{p^r}. \end{aligned}$$

By our induction hypothesis we must then have

$$\sum_{a=0}^{p^{r+1}-1} a^m \equiv 0 \pmod{p^r},$$

and the lemma follows. $\square$

LEMMA 3.2. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 0$. For all $h \in \mathbf{Z}$, $h \geq 1$,*

$$\frac{1}{q^h f_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) \left(\langle a + q\tau \rangle - 1\right)^n \equiv 0 \pmod{f_\chi^{-1} p^{-1} q^{n-1} \mathfrak{o}}.$$

Proof. This is obvious for $n = 0$ since writing

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \sum_{a=1}^{q^h f_\chi} \chi(a) - \sum_{a=1}^{p^{-1} q^h f_\chi} \chi(pa)$$

allows us to derive

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \begin{cases} p^{-1} q^h (p-1), & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

So let us assume that $n \geq 1$.

Let $h = 1$. Then $\langle a+q\tau\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for all $a \in \mathbf{Z}$ such that $(a,p) = 1$ implies that

$$\left(\langle a+q\tau\rangle - 1\right)^n \equiv 0 \pmod{q^n\mathfrak{o}},$$

and the lemma holds for this case.

Now assume that $h \geq 1$. We can rewrite our sum as follows:

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)(\langle a+q\tau\rangle - 1)^n = \sum_{v=0}^{q^{h-1}-1} \sum_{\substack{u=1\\(u+vqf_\chi,p)=1}}^{qf_\chi} \chi(u+vqf_\chi)(\langle u+vqf_\chi+q\tau\rangle - 1)^n.$$

Since $|\tau|_p \leq 1$, we can write

$$\begin{aligned}\langle u+vqf_\chi+q\tau\rangle &= \left(u+vqf_\chi+q\tau\right)\omega^{-1}\left(u+vqf_\chi+q\tau\right)\\ &= (u+q\tau)\,\omega^{-1}(u+q\tau) + vqf_\chi\omega^{-1}(u+q\tau)\\ &= \langle u+q\tau\rangle + vqf_\chi\omega^{-1}(u).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\left(\langle a+q\tau\rangle - 1\right)^n\\ &\qquad = \sum_{\substack{u=1\\(u,p)=1}}^{qf_\chi} \chi(u) \sum_{v=0}^{q^{h-1}-1} \left(\langle u+q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u)\right)^n.\end{aligned}$$

By expanding, the inner sum on the right can be written

$$\begin{aligned}&\sum_{v=0}^{q^{h-1}-1} (\langle u+q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u))^n\\ &\qquad = \sum_{k=0}^{n} \binom{n}{k} (\langle u+q\tau\rangle - 1)^{n-k} q^k f_\chi^k \omega^{-k}(u) \sum_{v=0}^{q^{h-1}-1} v^k.\end{aligned}$$

Since $(u,p) = 1$, we obtain the equivalence

$$q^k\left(\langle u+q\tau\rangle - 1\right)^{n-k} \equiv 0 \pmod{q^n\mathfrak{o}}$$

for each k, $0 \leq k \leq n$. Furthermore, by Lemma 3.1

$$\sum_{v=0}^{q^{h-1}-1} v^k \equiv 0 \pmod{p^{-1}q^{h-1}}$$

for each such k. Therefore

$$\sum_{v=0}^{q^{h-1}-1} \left(\langle u+q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u)\right)^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}}.$$

This implies that

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)(\langle a+q\tau\rangle - 1)^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}},$$

yielding the result. □

We now derive our bound on the magnitude of $c_n(\tau)$.

PROPOSITION 3.3. *For all* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and for* $n \in \mathbf{Z}$, $n \geq 0$, *we have* $|c_n(\tau)|_p \leq |pqf_\chi|_p^{-1}|q|_p^n$.

Proof. This follows in a manner similar to that given for the proof of the bound $|c_n(0)|_p \leq \left|q^2 f_\chi\right|_p^{-1}|q|_p^n$ found in [13] (Lemma 4 of Chapter 3). However, in this case we use Lemma 2.3 and the properties of χ and ω to derive

$$b_n(\tau) = \lim_{h\to\infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\langle a+q\tau\rangle^n$$

for each $n \geq 0$, and thus

$$c_n(\tau) = \lim_{h\to\infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\left(\langle a+q\tau\rangle - 1\right)^n$$

for each such n. From Lemma 3.2 we obtain

$$c_n(\tau) \equiv 0 \pmod{f_\chi^{-1}p^{-1}q^{n-1}\mathfrak{o}},$$

and thus the result. □

For our immediate concern we only need this proposition to hold for all $\tau \in \overline{\mathbf{Q}}_p$ such that $|\tau|_p \leq 1$. However, later on we shall need it in the form in which we have it.

We are now ready to begin the construction of our L-function.

THEOREM 3.4. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a power series* $A_\chi(s,\tau)$ *in* $K_\tau[[s]]$ *such that the power series converges on* $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, *and for each* $n \in \mathbf{Z}$, $n \geq 0$, $A_\chi(n,\tau)$ *satisfies*

$$A_\chi(n,\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right) .$$

Proof. By Proposition 3.3, $|c_n(\tau)|_p \leq C|q|_p^n$ for all $n \geq 0$, where $C = |pqf_\chi|_p^{-1}$. Therefore we can apply Theorem 2.7 to the sequences $\{b_n(\tau)\}_{n=0}^\infty$ and $\{c_n(\tau)\}_{n=0}^\infty$ in $K_\tau = \mathbf{Q}_p(\chi,\tau)$, and for $\rho = |q|_p < |p|_p^{1/(p-1)}$, yielding this result. □

Let us denote $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$.

THEOREM 3.5. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a unique* p*-adic, meromorphic function* $L_p(s,\tau;\chi)$ *that can be expressed in the form*

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^\infty a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *having coefficients* $a_n(\tau) \in \mathbf{Q}_p(\chi,\tau)$, *with*

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

Furthermore, for each $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right) .$$

Proof. Let

(13) $$L_p(s,\tau;\chi) = \frac{1}{s-1}A_\chi(1-s,\tau)$$

with the $A_\chi(s,\tau)$ as in Theorem 3.4. Then from the properties of $A_\chi(s,\tau)$, the power series must converge in the given domain, and for $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}A_\chi(n,\tau) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right) .$$

Note that

$$\begin{aligned} a_{-1}(\tau) &= A_\chi(0,\tau) = B_{0,\chi}(q\tau) - \chi(p)p^{-1}B_{0,\chi}\left(p^{-1}q\tau\right) \\ &= (1-\chi(p)p^{-1})B_{0,\chi} , \end{aligned}$$

and thus

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

The uniqueness of $L_p(s, \tau; \chi)$ follows from Lemma 2.5. $\square$

At this point we have not completed our goal of showing that the p-adic function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. In order to prove this, we will need to study the coefficients, $a_n(\tau)$, of the power series expansion of $L_p(s, \tau; \chi)$ for each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. From the results of this we will show that the function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for any sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to τ, the values $L_p(1-n, \tau_i; \chi)$ converge to $L_p(1-n, \tau; \chi)$ for each $n \in \mathbf{Z}$, $n \geq 1$.

3.2 $L_p(s, \tau; \chi)$ FOR $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$

Our previous work has been for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. To extend this result to all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we need to find a way to express $a_n(\tau)$ so that it can be defined for these values of τ.

For $k \in \mathbf{Z}$, $k \geq 0$, the Stirling numbers of the first kind, $s(n,k)$, are defined by the generating function

$$\sum_{n=0}^{\infty} s(n,k) \frac{t^n}{n!} = \frac{1}{k!} (\log(1+t))^k . \tag{14}$$

Since the power series expansion of $\log(1+t)$ lacks a constant term, we must have $s(n,k) = 0$ whenever $0 \leq n < k$. We also have $s(n,n) = 1$ for all $n \geq 0$. The $s(n,k)$ are integers, where $n, k \in \mathbf{Z}$, $n \geq 0$, $k \geq 0$, and they satisfy the relation

$$\binom{x}{n} = \frac{1}{n!} \sum_{k=0}^{n} s(n,k) x^k . \tag{15}$$

For additional information on Stirling numbers of the first kind we refer the reader to [6], pp. 214–217.

LEMMA 3.6. *Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$,*

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau) .$$

Proof. From Corollary 2.8 we can write

$$A_\chi(s,\tau)=\sum_{m=0}^{\infty}\binom{s}{m}c_m(\tau),$$

where $s\in\mathbf{C}_p$ such that $|s|_p<|p|_p^{1/(p-1)}|q|_p^{-1}$. Now, expanding the quantity $\binom{s}{m}$ according to (15) yields

$$A_\chi(s,\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}s(m,n)c_m(\tau)s^n,$$

where $s(m,n)\in\mathbf{Z}$ is a Stirling number of the first kind. At this point we wish to switch the order of summation in this expression, but before doing so we must show that the terms in the summation converge to 0 at a sufficient rate.

Let $\epsilon>0$ and let $\xi\in\mathbf{C}_p$ such that $|\xi|_p<|p|_p^{1/(p-1)}|q|_p^{-1}$. Then there exists $\delta\in\mathbf{R}$, $0\leq\delta<1$, such that

$$|\xi|_p=\delta\cdot|p|_p^{1/(p-1)}|q|_p^{-1}.$$

Let $N,M\in\mathbf{Z}$, $N>0$, $M>0$, such that if $n\geq N$ then $|pqf_\chi|_p^{-1}\delta^n<\epsilon$, and if $m\geq M$ then $|pqf_\chi|_p^{-1}|p|_p^{-m/(p-1)}|q|_p^m<\epsilon$ (such an M exists since $0\leq|p|_p^{-1/(p-1)}|q|_p<1$).

Let $m,n\in\mathbf{Z}$, $m\geq0$, $n\geq0$. If $n>m$, then $s(m,n)=0$, and so

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p=0.$$

Thus we can assume that $m=\max\{m,n\}$. Consider

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p\leq|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n.$$

Utilizing Proposition 3.3 and the fact that $v_p(m!)\leq m/(p-1)$, we can write

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n\leq|pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n}\delta^n.$$

Suppose that $m\geq M+N$. If $m-n<M$, then

$$M+N\leq m<M+n,$$

so that $n>N$. Thus

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n\leq|pqf_\chi|_p^{-1}\delta^n<\epsilon.$$

If $m-n\geq M$, then

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n \leq |pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n} < \epsilon\,.$$

Either case implies that

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p < \epsilon\,.$$

Therefore, whenever $\max\{m,n\} \geq M+N$, this bound must hold, implying that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n,$$

by Proposition 2.4.

Writing

$$A_\chi(s,\tau) = \sum_{n=0}^{\infty}s^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau)\,,$$

we have from (13),

$$\begin{aligned} L_p(s,\tau;\chi) &= \frac{1}{s-1}\sum_{n=0}^{\infty}(1-s)^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau) \\ &= \sum_{n=-1}^{\infty}(-1)^{n+1}(s-1)^n\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau)\,, \end{aligned}$$

which implies the lemma, since we must have convergence for the inner sum. □

Since we have only derived $L_p(s,\tau;\chi)$ for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, we cannot say that $a_n(\tau)$ is defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$, let us define

$$a_n(\tau) = (-1)^{n+1}\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau)\,, \tag{16}$$

for these values of τ. Note that in the proof of Lemma 3.6, the only influence generated by the value of τ is in the bound of the value of $|c_m(\tau)|_p$, which was determined in Proposition 3.3. However, this proposition holds for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Thus this sum converges and $a_n(\tau)$ is well-defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

THEOREM 3.7. *Let* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and let* $\{\tau_i\}_{i=1}^{\infty}$ *be a sequence in* $\overline{\mathbf{Q}}_p$, *with* $|\tau_i|_p \leq 1$, *such that* $\tau_i \to \tau$. *Then for* $n \in \mathbf{Z}$, $n \geq -1$,

$$\lim_{i \to \infty} a_n(\tau_i) = a_n(\tau).$$

Proof. By definition, for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for $n \in \mathbf{Z}$, $n \geq -1$, we have the expansion

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau),$$

and as we have seen, regardless of the value of τ,

$$\left| \frac{1}{m!} s(m, n+1) c_m(\tau) \right|_p \leq |pqf_\chi|_p^{-1} |p|_p^{-m/(p-1)} |q|_p^m \to 0$$

as $m \to \infty$. Therefore given $\epsilon > 0$ there must exist some $m_0 \in \mathbf{Z}$, $m_0 \geq n+1$, such that

$$\left| \sum_{m=m_0+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau) \right|_p < \epsilon.$$

Thus for any sequence $\{\tau_i\}_{i=1}^{\infty}$ in $\mathbf{Q}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$,

$$|a_n(\tau) - a_n(\tau_i)|_p \leq \max_{n+1 \leq m \leq m_0} \left\{ \epsilon, \left| \frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right) \right|_p \right\}.$$

Since $\tau_i \to \tau$ and $c_m(\tau)$ is a polynomial in τ, we see that

$$\left| \frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right) \right|_p < \epsilon$$

for all m with $n+1 \leq m \leq m_0$ when i is sufficiently large, which implies that

$$|a_n(\tau) - a_n(\tau_i)|_p < \epsilon$$

for such i. Therefore the theorem must hold. $\square$

The purpose of the following three lemmas is to build an upper bound for the value of $|a_n(\tau)|_p$. After doing so we can define $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

LEMMA 3.8. *Let p be prime. If $i, n \in \mathbf{Z}$ with $1 \leq i \leq n$, then*

$$\left|\binom{n}{i}p^i\right|_p \leq |np|_p .$$

Proof. For $i \in \mathbf{Z}$ such that $1 \leq i \leq n$, (8) implies that $v_p(i!) \leq i-1$, or equivalently, $|i!|_p \geq |p|_p^{i-1}$. Therefore by combining this with

$$\left|\binom{n}{i}p^i\right|_p = \left|\frac{n(n-1)\cdots(n-i+1)}{i!}\right|_p |p|_p^i \leq |i!|_p^{-1}|n|_p|p|_p^i ,$$

the result will follow. □

LEMMA 3.9. *Let p be prime. Then for $m, n \in \mathbf{Z}$, $m > n \geq 0$,*

$$\left|\frac{n!}{m!}s(m,n)q^m\right|_p \leq |np|_p|q|_p^n .$$

Proof. From (14), the generating function for the $s(m,n)$, we obtain

$$\sum_{m=0}^{\infty}\frac{n!}{m!}s(m,n)q^m t^m = (\log(1+qt))^n .$$

Thus we wish to evaluate the power of p that divides the coefficient of t^m in the expansion of $(\log(1+qt))^n$. The power series expansion of the logarithm function (10) yields

$$(\log(1+qt))^n = \left(\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}q^i t^i\right)^n ,$$

and by factoring qt out of the sum,

$$(\log(1+qt))^n = q^n t^n\left(1 + pt\sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{i}p^{-1}q^{i-1}t^{i-2}\right)^n .$$

For $i \geq 2$, we see that $p^{-1}q^{i-1}/i \in \mathbf{Z}_p$. Therefore

$$(\log(1+qt))^n = q^n t^n(1+ptf(t))^n,$$

where $f(t) \in \mathbf{Z}_p[[t]]$. Now, this can be written

$$(\log(1+qt))^n = q^n t^n + q^n t^n\sum_{i=1}^{n}\binom{n}{i}p^i t^i f(t)^i,$$

and from Lemma 3.8, the p-adic absolute value of the coefficients of the terms in the sum on the right must be bounded above by $|np|_p|q|_p^n$. Thus, for $m > n$, the coefficient of t^m must also be bounded above by this quantity, implying the result. □

LEMMA 3.10. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Then for $n \in \mathbf{Z}$, $n \geq 0$,*

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}.$$

Proof. From (16), we see that for $n \in \mathbf{Z}$, $n \geq 0$,

$$f_\chi n! a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{n!}{m!} f_\chi s(m, n+1) c_m(\tau).$$

Proposition 3.3 implies that

$$f_\chi c_m(\tau) \equiv 0 \pmod{p^{-1} q^{m-1} \mathfrak{o}}.$$

By Lemma 3.9, when $m \geq n+2$,

$$\frac{n!}{m!} s(m, n+1) \equiv 0 \pmod{p q^{n-m+1} \mathfrak{o}}.$$

Thus

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}. \qquad \square$$

We are nearing our goal of defining $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. The final step before doing so is proving the following lemma on the convergence of a specific infinite sum.

LEMMA 3.11. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Then the sum*

$$\sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

converges for all $s \in \mathfrak{D}$.

Proof. Let $\xi \in \mathfrak{D}$. Then $|\xi - 1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$. Thus there must be some $\delta \in \mathbf{R}$, $0 \leq \delta < 1$, such that

$$|\xi - 1|_p = \delta \cdot |p|_p^{1/(p-1)} |q|_p^{-1}.$$

Let $n \in \mathbf{Z}$, $n \geq 0$. From Lemma 3.10

$$f_\chi (n+1)! a_n(\tau) \equiv (-1)^{n+1} f_\chi c_{n+1}(\tau) \pmod{(n+1) q^n \mathfrak{o}},$$

and from Proposition 3.3,

$$|f_\chi c_{n+1}(\tau)|_p \leq |p|_p^{-1} |q|_p^n.$$

Therefore

$$\left|f_\chi(n+1)!a_n(\tau)\right|_p \leq |p|_p^{-1}|q|_p^n,$$

which implies that

$$\left|a_n(\tau)\right|_p \leq \left|f_\chi(n+1)!p\right|_p^{-1}|q|_p^n.$$

Thus

$$\left|a_n(\tau)(\xi-1)^n\right|_p \leq \left|f_\chi(n+1)!p\right|_p^{-1}|p|_p^{n/(p-1)}\delta^n.$$

Now,

$$v_p((n+1)!) \leq \frac{n}{p-1},$$

so that

$$\left|a_n(\tau)(\xi-1)^n\right|_p \leq \left|f_\chi p\right|_p^{-1}\delta^n.$$

Since $0 \leq \delta < 1$, we see that $\left|a_n(\tau)(\xi-1)^n\right|_p \to 0$ as $n \to \infty$. Thus the sum

$$\sum_{n=0}^{\infty} a_n(\tau)(\xi-1)^n$$

must converge. □

Note that from this proof we have obtained the bound

$$\left|a_n(\tau)\right|_p \leq \left|f_\chi(n+1)!p\right|_p^{-1}|q|_p^n, \tag{17}$$

for each $n \in \mathbf{Z}$, $n \geq -1$, and for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

Now let us define

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$. This definition is consistent with what we already have for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. We will now show that, for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, this function satisfies

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. To do this, we prove the following:

LEMMA 3.12. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Then for each $n \in \mathbf{Z}$, $n \geq 1$,*

$$\lim_{i\to\infty} L_p(1-n, \tau_i; \chi) = L_p(1-n, \tau; \chi).$$

Proof. We can write

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{m=0}^{\infty} a_m(\tau)(s-1)^m,$$

where the power series converges for each $s \in \mathfrak{D}$.

Let $\epsilon > 0$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then we must have $1 - n \in \mathfrak{D}$, and thus the power series converges for $s = 1 - n$. Also, by (17)

$$|a_m(\tau)(-n)^m|_p \leq |f_\chi(m+1)!p|_p^{-1}|nq|_p^m \to 0$$

independently of τ as $m \to \infty$. Therefore, for $m_0 \in \mathbf{Z}$ sufficiently large,

$$\left|\sum_{m=m_0}^{\infty} a_m(\tau)(-n)^m\right|_p < \epsilon.$$

For $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, let $\{\tau_i\}_{i=1}^{\infty}$ be in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Consider

$$|L_p(1-n, \tau; \chi) - L_p(1-n, \tau_i; \chi)|_p \leq \max_{0 \leq m < m_0} \left\{\epsilon, |(a_m(\tau) - a_m(\tau_i))(-n)^m|_p\right\}.$$

Since $a_m(\tau_i) \to a_m(\tau)$ as $\tau_i \to \tau$, we have

$$|L_p(1-n, \tau; \chi) - L_p(1-n, \tau_i; \chi)|_p < \epsilon$$

for i sufficiently large. Thus the lemma must hold. $\square$

At this point we have finally proven

THEOREM 3.13. *For each $\tau \in \mathbf{C}_p$, with $|\tau|_p \leq 1$, there exists a unique p-adic, meromorphic function $L_p(s, \tau; \chi)$ that satisfies*

$$L_p(1-n, \tau; \chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, this function can be expressed in the form

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *and*

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1\,. \end{cases} \qquad \square$$

Since $L_p(s,\tau;\chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \le 1$, we now have a p-adic function of two variables, $L_p(s,t;\chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \le 1$.

4. PROPERTIES OF $L_p(s,t;\chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ.

4.1 A SYMMETRY PROPERTY IN t

The first property we obtain regarding $L_p(s,t;\chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s,t;\chi)$ interpolates the values

$$L_p(1-n,t;\chi) = -\frac{1}{n} b_n(t)\,, \tag{18}$$

for $n \in \mathbf{Z}$, $n \ge 1$, and $t \in \mathbf{C}_p$, $|t|_p \le 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right), \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t)\,. \tag{20}$$

LEMMA 4.1. *For all* $n \in \mathbf{Z}$, $n \ge 0$, *we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1}\,.$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \ge 1$. Because $B_{n,1} = 0$ for odd $n \ge 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + nB_{1,1}t^{n-1}\,.$$