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LEMMA 2.3. Let  $\tau \in \mathbf{C}_p$ . In the field  $\mathbf{Q}_p(\chi, \tau)$ , for all  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n.$$

*Proof.* By applying Lemma 2.2 to (4), we obtain

$$B_{n, \chi} = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi(a)a^n.$$

Therefore, by (2),

$$\begin{aligned} B_{n, \chi_n}(\tau) &= \sum_{m=0}^n \binom{n}{m} \tau^{n-m} \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a) \sum_{m=0}^n \binom{n}{m} \tau^{n-m} a^m. \end{aligned}$$

Since  $f_\chi$  and  $f_{\chi_n}$  differ by a factor that is a power of  $p$ , we must have

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n,$$

and the proof is complete.  $\square$

## 2.5 $p$ -ADIC FUNCTIONS

Let  $K$  be an extension of  $\mathbf{Q}_p$  contained in  $\mathbf{C}_p$ . An infinite series  $\sum_{n=0}^{\infty} a_n$ ,  $a_n \in K$ , converges in  $K$  if and only if  $|a_n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $K[[x]]$  be the algebra of formal power series in  $x$ . Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in  $K[[x]]$ , converges at  $x = \xi$ ,  $\xi \in \mathbf{C}_p$ , if and only if  $|a_n \xi^n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore whenever a power series  $A(x)$  converges at some  $\xi_0 \in \mathbf{C}_p$ , then it must converge at all  $\xi \in \mathbf{C}_p$  such that  $|\xi|_p \leq |\xi_0|_p$ . The following result, for double series in  $K$ , can be found in [8].

PROPOSITION 2.4. Let  $b_{n,m} \in K$ , and suppose that for each  $\epsilon > 0$  there exists  $N \in \mathbf{Z}$ , depending on  $\epsilon$ , such that if  $\max\{n,m\} \geq N$ , then  $|b_{n,m}|_p \leq \epsilon$ . Then both series

$$\sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} b_{n,m} \right) \quad \text{and} \quad \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} b_{n,m} \right)$$

converge, and their sums are equal.

There are two power series that we wish to make note of in particular. First we define the  $p$ -adic exponential function,  $\exp(x)$ , in  $\mathbf{Q}_p[[x]]$ , by

$$(9) \quad \exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

From (8) we can conclude that this power series converges in  $\{x \in \mathbf{C}_p : |x|_p < p^{-1/(p-1)}\}$ . The  $p$ -adic logarithm function,  $\log(x)$ , in  $\mathbf{Q}_p[[x]]$ , is defined by

$$(10) \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n,$$

the power series converging in the domain  $\{x \in \mathbf{C}_p : |x|_p < 1\}$ . For  $|x|_p < p^{-1/(p-1)}$ , we have  $\log(\exp(x)) = x$  and  $\exp(\log(1+x)) = 1+x$ .

The following property is a uniqueness property for power series, found in [13].

LEMMA 2.5. Let  $A(x), B(x) \in K[[x]]$ , such that each converges in a neighborhood of 0 in  $\mathbf{C}_p$ . If  $A(\xi_n) = B(\xi_n)$  for a sequence  $\{\xi_n\}_{n=0}^{\infty}$ ,  $\xi_n \neq 0$ , in  $\mathbf{C}_p$ , such that  $\xi_n \rightarrow 0$ , then  $A(x) = B(x)$ .

Let  $U$  be an open subset of  $\mathbf{C}_p$ , contained in the domain of the  $p$ -adic function  $f$ . We say that  $f$  is differentiable at  $x \in U$  if the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists. If this limit exists for each  $x \in U$ , then we say that  $f$  is differentiable in  $U$ .

The relationship between the derivatives of a function and its power series expansion is given in the following result, found in [8].

PROPOSITION 2.6. Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with coefficients in  $\mathbf{C}_p$ , and suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - \alpha)^n$$

converges on some closed ball  $B$  in  $\mathbf{C}_p$ . Then

i) For each  $x \in B$ , the  $k^{\text{th}}$  derivative  $f^{(k)}(x)$  exists, and is given by

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x - \alpha)^{n-k},$$

and we have

$$a_k = \frac{1}{k!} f^{(k)}(\alpha).$$

ii) Let  $\beta \in B$ . Then there exists a series  $\sum_{n=0}^{\infty} b_n x^n$  such that

$$f(x) = \sum_{n=0}^{\infty} b_n (x - \beta)^n$$

for any  $x \in B$ . Both series  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=0}^{\infty} b_n x^n$  have the same region of convergence.

Now let  $K$  be a finite extension of  $\mathbf{Q}_p$ . For  $A(x) \in K[[x]]$ ,  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ , where  $a_n \in K$ , define

$$\|A\| = \sup_n |a_n|_p.$$

Let  $P_K = \{A(x) \in K[[x]] : \|A\| < \infty\}$ . Then  $\|\cdot\|$  defines a norm on  $P_K$ , and so  $K[x] \subset P_K \subset K[[x]]$ . Furthermore  $P_K$  is complete in this norm.

Let  $\{b_n\}_{n=0}^{\infty}$  be a sequence of elements of  $K$ , and let the sequence  $\{c_n\}_{n=0}^{\infty}$  be defined by

$$(11) \quad c_n = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} b_m$$

for each  $n \in \mathbf{Z}$ ,  $n \geq 0$ . Then  $c_n \in K$  for each  $n \geq 0$ . Note that (11) implies that these sequences must satisfy

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = e^{-t} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$

This implies that

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and so we have the relationship

$$(12) \quad b_n = \sum_{m=0}^n \binom{n}{m} c_m$$

for each  $n \in \mathbf{Z}$ ,  $n \geq 0$ . We can reverse this process to derive (11) given (12). Thus (11) and (12) must be equivalent. The following relate to sequences that satisfy (11) and (12), and are found in [13].

**THEOREM 2.7.** *Let  $\{b_n\}_{n=0}^{\infty}$  and  $\{c_n\}_{n=0}^{\infty}$  be defined as in the above relation. Let  $\rho \in \mathbf{R}$  such that  $0 < \rho < |p|_p^{1/(p-1)}$ . If  $|c_n|_p \leq C\rho^n$  for all  $n \geq 0$ , where  $C > 0$ , then there exists a unique power series  $A(x) \in P_K$  such that  $A(x)$  converges at every  $\xi \in \mathbf{C}_p$  with  $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$ , and  $A(n) = b_n$  for every  $n \geq 0$ .*

**COROLLARY 2.8.** *Let  $A(x)$  be the power series from the theorem. Then for each  $\xi \in \mathbf{C}_p$  such that  $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$ , we have*

$$A(\xi) = \sum_{n=0}^{\infty} c_n \binom{\xi}{n}.$$

Theorem 2.7 can be applied to the sequence  $\{b_n\}_{n=0}^{\infty}$  in  $K = \mathbf{Q}_p(\chi)$ , where

$$b_n = (1 - \chi_n(p)p^{n-1})B_{n,\chi_n},$$

in order to obtain a power series  $A_\chi(s)$  satisfying  $A_\chi(n) = b_n$ , and converging on the domain  $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$ . (Since  $|p|_p^{1/(p-1)}|q|_p^{-1} > 1$  and  $|n|_p \leq 1$  for each  $n \in \mathbf{Z}$ , all of  $\mathbf{Z}$  is contained in this domain.) From this a  $p$ -adic function,  $L_p(s; \chi)$ , can be derived that interpolates the values

$$L_p(1 - n; \chi) = -\frac{1}{n}b_n,$$

and which converges in  $\{s \in \mathbf{C}_p : |s - 1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$ , except  $s \neq 1$  if  $\chi = 1$ . Note that if  $\chi$  is odd, then  $\chi_n$  is even when  $n$  is odd, and  $\chi_n$  is odd when  $n$  is even. Thus the quantity  $(1 - \chi_n(p)p^{n-1})B_{n,\chi_n} = 0$  for all  $n \in \mathbf{Z}$ ,  $n \geq 1$ , as we saw from the properties of generalized Bernoulli numbers. Therefore  $L_p(s; \chi)$  vanishes on a sequence such as  $\{-p^m\}_{m=0}^{\infty}$ , which has 0 as a limit point, implying that for such  $\chi$  we must have  $L_p(s; \chi) \equiv 0$ .