

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: p-ADIC L-FUNCTION OF TWO VARIABLES
Autor: Fox, Glenn J.
Kapitel: 2.4 The p-adic number field
DOI: <https://doi.org/10.5169/seals-64800>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2.4 THE p -ADIC NUMBER FIELD

Let p be prime. We shall use \mathbf{Z}_p to represent the p -adic integers, and \mathbf{Q}_p the p -adic rationals. Let $|\cdot|_p$ denote the p -adic absolute value on \mathbf{Q}_p , normalized so that $|p|_p = p^{-1}$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of \mathbf{Q}_p . The absolute value on \mathbf{Q}_p extends uniquely to $\overline{\mathbf{Q}}_p$, however $\overline{\mathbf{Q}}_p$ is not complete with respect to the absolute value. Let \mathbf{C}_p be the completion of $\overline{\mathbf{Q}}_p$ with respect to this absolute value. Then the absolute value extends to \mathbf{C}_p , and $\overline{\mathbf{Q}}_p$ is dense in \mathbf{C}_p . We also have \mathbf{C}_p algebraically closed. Furthermore, on \mathbf{C}_p the absolute value is non-Archimedean, and so

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

for any $a, b \in \mathbf{C}_p$. Note that the two fields \mathbf{C} and \mathbf{C}_p are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of \mathbf{C}_p in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \quad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then \mathfrak{p} is a maximal ideal of \mathfrak{o} . If $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq |p|_p^s$, where $s \in \mathbf{Q}$, then $\tau \in p^s \mathfrak{o}$, and so we shall also write this as $\tau \equiv 0 \pmod{p^s \mathfrak{o}}$.

Any $n \in \mathbf{Z}$, $n > 0$, can be uniquely expressed in the form $n = \sum_{m=0}^k a_m p^m$, where $a_m \in \mathbf{Z}$, $0 \leq a_m \leq p - 1$, for $m = 0, 1, \dots, k$, and $a_k \neq 0$. For such n , we define

$$s_p(n) = \sum_{m=0}^k a_m,$$

the sum of the p -adic digits of n , and also define $s_p(0) = 0$. For any $n \in \mathbf{Z}$, let $v_p(n)$ be the highest power of p dividing n . This function is additive, and relates to the function $s_p(n)$ by means of the identity

$$(8) \quad v_p(n!) = \frac{n - s_p(n)}{p - 1},$$

which holds for all $n \geq 0$. Note that for $n \geq 1$ this implies that

$$v_p(n!) \leq \frac{n - 1}{p - 1}.$$

The definition of this function can be extended to all of \mathbf{Q} by taking $v_p(1/n) = -v_p(n)$.

Throughout we let $q = 4$ if $p = 2$, and $q = p$ otherwise. Note that there exist $\phi(q)$ distinct solutions, modulo q , to the equation $x^{\phi(q)} - 1 = 0$, and each solution must be congruent to one of the values $a \in \mathbf{Z}$, where $1 \leq a \leq q$,

$(a, p) = 1$. Thus, by Hensel's Lemma, given $a \in \mathbf{Z}$ with $(a, p) = 1$, there exists a unique $\omega(a) \in \mathbf{Z}_p$, where $\omega(a)^{\phi(q)} = 1$, such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting $\omega(a) = 0$ for $a \in \mathbf{Z}$ such that $(a, p) \neq 1$, we see that ω is actually a Dirichlet character, called the Teichmüller character, having conductor $f_\omega = q$. Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$. For $p \geq 3$, $\lim_{n \rightarrow \infty} a^{p^n} = \omega(a)$, since $a^{p^n} \equiv a \pmod{p}$ and $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$.

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character ω . If $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbf{Z}$, $a + qt \equiv a \pmod{q\mathfrak{o}}$. Thus we define

$$\omega(a + qt) = \omega(a)$$

for these values of t . We also define

$$\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$$

for such t .

Fix an embedding of the algebraic closure of \mathbf{Q} , $\overline{\mathbf{Q}}$, into \mathbf{C}_p . We may then consider the values of a Dirichlet character χ as lying in \mathbf{C}_p . For $n \in \mathbf{Z}$ we define the product $\chi_n = \chi\omega^{-n}$ in the sense of the product of characters. This implies that $f_{\chi_n} \mid f_\chi q$. However, since we can write $\chi = \chi_n\omega^n$, we also have $f_\chi \mid f_{\chi_n} q$. Thus f_χ and f_{χ_n} differ by a factor that is a power of p . In fact, either $f_{\chi_n}/f_\chi \in \mathbf{Z}$ and divides q , or $f_\chi/f_{\chi_n} \in \mathbf{Z}$ and divides q .

Let $\mathbf{Q}_p(\chi)$ denote the field generated over \mathbf{Q}_p by all values $\chi(a)$, $a \in \mathbf{Z}$. In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field $\mathbf{Q}_p(\chi)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} (B_{n+1,\chi}(p^h f_\chi) - B_{n+1,\chi}(0)).$$

From this we can obtain

LEMMA 2.3. Let $\tau \in \mathbf{C}_p$. In the field $\mathbf{Q}_p(\chi, \tau)$, for all $n \in \mathbf{Z}$, $n \geq 0$,

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n.$$

Proof. By applying Lemma 2.2 to (4), we obtain

$$B_{n, \chi} = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi(a)a^n.$$

Therefore, by (2),

$$\begin{aligned} B_{n, \chi_n}(\tau) &= \sum_{m=0}^n \binom{n}{m} \tau^{n-m} \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a) \sum_{m=0}^n \binom{n}{m} \tau^{n-m} a^m. \end{aligned}$$

Since f_χ and f_{χ_n} differ by a factor that is a power of p , we must have

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n,$$

and the proof is complete. \square

2.5 p -ADIC FUNCTIONS

Let K be an extension of \mathbf{Q}_p contained in \mathbf{C}_p . An infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in K$, converges in K if and only if $|a_n|_p \rightarrow 0$ as $n \rightarrow \infty$. Let $K[[x]]$ be the algebra of formal power series in x . Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in $K[[x]]$, converges at $x = \xi$, $\xi \in \mathbf{C}_p$, if and only if $|a_n \xi^n|_p \rightarrow 0$ as $n \rightarrow \infty$. Therefore whenever a power series $A(x)$ converges at some $\xi_0 \in \mathbf{C}_p$, then it must converge at all $\xi \in \mathbf{C}_p$ such that $|\xi|_p \leq |\xi_0|_p$. The following result, for double series in K , can be found in [8].