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**Autor:** Fox, Glenn J.  
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Since any Dirichlet character  $\chi$  is multiplicative, we must have  $\chi(-1) = \pm 1$ . A character  $\chi$  is said to be odd if  $\chi(-1) = -1$ , and even if  $\chi(-1) = 1$ .

### 2.2 GENERALIZED BERNOULLI POLYNOMIALS

Let  $\chi$  be a Dirichlet character with conductor  $f_\chi$ . Then we define the functions,  $B_{n,\chi}(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by the generating function

$$(1) \quad \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with  $\chi$ ,  $B_{n,\chi}$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that  $B_{n,\chi}(0) = B_{n,\chi}$ . Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!},$$

and from this we obtain

$$(2) \quad B_{n,\chi}(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions  $B_{n,\chi}(t)$ , defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with  $\chi$ . Let  $\mathbf{Z}[\chi]$  denote the ring generated over  $\mathbf{Z}$  by all the values  $\chi(a)$ ,  $a \in \mathbf{Z}$ , and  $\mathbf{Q}(\chi)$  the field generated over  $\mathbf{Q}$  by all such values. Then it can be shown that  $f_\chi B_{n,\chi}$  must be in  $\mathbf{Z}[\chi]$  for each  $n \geq 0$  whenever  $\chi \neq 1$ . In general, we have  $B_{n,\chi} \in \mathbf{Q}(\chi)$  for each  $n \geq 0$ , and so  $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$ . The polynomials  $B_{n,\chi}(t)$  exhibit the property that, for all  $n \geq 0$ ,

$$(3) \quad B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever  $\chi \neq 1$ . Thus  $B_{n,\chi}(t)$ , for  $\chi \neq 1$ , is either an even function or an odd function according to whether  $(-1)^n \chi(-1)$  is 1 or  $-1$ . From (3) we obtain

$$B_{n,\chi} = (-1)^n \chi(-1) B_{n,\chi},$$

and so  $B_{n,\chi} = 0$  whenever  $n$  is even and  $\chi$  is odd, or whenever  $n$  is odd and  $\chi$  is even,  $\chi \neq 1$ . Another property that the polynomials satisfy is that for  $m \in \mathbf{Z}$ ,  $m \geq 1$ ,

$$(4) \quad B_{n,\chi}(mf_\chi + t) - B_{n,\chi}(t) = n \sum_{a=1}^{mf_\chi} \chi(a)(a+t)^{n-1},$$

for all  $n \geq 0$ . This can be derived from (1). Note that for  $\chi = 1$  and  $t = 0$  this becomes

$$\frac{1}{n} (B_{n,1}(m) - B_{n,1}) = \sum_{a=1}^m a^{n-1}.$$

If  $\chi \neq 1$ , then it can be shown that  $\sum_{a=1}^{f_\chi} \chi(a) = 0$ , and from the above relations we can derive

$$B_{0,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a)$$

for all  $\chi$ . Therefore

$$B_{0,\chi} = \begin{cases} 0, & \text{if } \chi \neq 1 \\ 1, & \text{if } \chi = 1. \end{cases}$$

The ordinary Bernoulli polynomials,  $B_n(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , are defined by

$$(5) \quad \frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and the Bernoulli numbers,  $B_n$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi.$$

From this we obtain the values  $B_0 = 1$ ,  $B_1 = -1/2$ ,  $B_2 = 1/6$ ,  $B_4 = -1/30$ , ..., with  $B_n = 0$  for odd  $n \geq 3$ . For even  $n \geq 2$ , we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m.$$

Note that we again have the relations  $B_n(0) = B_n$  and

$$B_n(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m} t^m,$$

as we did for the generalized Bernoulli polynomials.

Some of the more important properties of Bernoulli polynomials are that

$$(6) \quad B_n(t + 1) - B_n(t) = nt^{n-1}$$

for all  $n \geq 1$ , and

$$B_n(1 - t) = (-1)^n B_n(t)$$

for  $n \geq 0$ . Each of these results can be derived from the generating function (5) above.

Similar to (4) for the generalized Bernoulli polynomials, whenever  $m, n \in \mathbf{Z}$ ,  $m \geq 1$ ,  $n \geq 1$ ,

$$\frac{1}{n} (B_n(m) - B_n) = \sum_{a=0}^{m-1} a^{n-1},$$

where we take  $0^0$  to be 1 in the case of  $a = 0$  and  $n = 1$ . Note that this can be derived from (6) since

$$B_n(m) - B_n = \sum_{a=0}^{m-1} (B_n(a + 1) - B_n(a)).$$

The Bernoulli numbers are rational numbers, and, in fact, the von Staudt-Clausen theorem states that for even  $n \geq 2$ ,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus the denominator of each  $B_n$  must be square-free.

The ordinary Bernoulli numbers are related to the generalized Bernoulli numbers in that for  $\chi = 1$  we have

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n,1} \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and since

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1},$$

we see that  $B_{n,1} = B_n$  for all  $n \neq 1$ , and  $B_{1,1} = -B_1$ . In fact, this can be written as  $B_{n,1} = (-1)^n B_n$ , and for the polynomials,  $B_{n,1}(t) = (-1)^n B_n(-t)$ .