

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** p-ADIC L-FUNCTION OF TWO VARIABLES  
**Autor:** Fox, Glenn J.  
**Kapitel:** 2.1 Dirichlet characters  
**DOI:** <https://doi.org/10.5169/seals-64800>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 25.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

Bernoulli polynomials in  $\tau$  for similar values of the variable  $s$ . These functions are designed so that  $L_p(s, 0; \chi) = L_p(s; \chi)$ . The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those  $\tau \in \overline{\mathbf{Q}}_p$  with  $|\tau|_p \leq 1$ . To complete the derivation we show that there exist functions  $L_p(s, \tau; \chi)$  for all  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , such that for every sequence  $\{\tau_i\}_{i=0}^\infty$  in  $\overline{\mathbf{Q}}_p$ , with  $|\tau_i|_p \leq 1$ , converging to some  $\tau \in \mathbf{C}_p$ , the sequence  $\{L_p(1-n, \tau_i; \chi)\}_{i=0}^\infty$ , with  $n \in \mathbf{Z}$ ,  $n \geq 1$ , converges to  $L_p(1-n, \tau; \chi)$ . Thus for each  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , the function  $L_p(s, \tau; \chi)$  must interpolate the appropriate expressions involving generalized Bernoulli polynomials for  $s = 1 - n$ ,  $n \in \mathbf{Z}$ ,  $n \geq 1$ .

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

## 2.1 DIRICHLET CHARACTERS

For  $n \in \mathbf{Z}$ ,  $n \geq 1$ , a Dirichlet character to the modulus  $n$  is a multiplicative map  $\chi : \mathbf{Z} \rightarrow \mathbf{C}$  such that  $\chi(a+n) = \chi(a)$  for all  $a \in \mathbf{Z}$ , and  $\chi(a) = 0$  if and only if  $(a, n) \neq 1$ . Since  $a^{\phi(n)} \equiv 1 \pmod{n}$  for all  $a$  such that  $(a, n) = 1$ ,  $\chi(a)$  must be a root of unity for such  $a$ .

If  $\chi$  is a Dirichlet character to the modulus  $n$ , then for any positive multiple  $m$  of  $n$  we can induce a Dirichlet character  $\psi$  to the modulus  $m$  according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a, m) = 1 \\ 0, & \text{if } (a, m) \neq 1. \end{cases}$$

The minimum modulus  $n$  for which a character  $\chi$  cannot be induced from some character to the modulus  $m$ ,  $m < n$ , is called the conductor of  $\chi$ , denoted  $f_\chi$ . We shall assume that each  $\chi$  is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters  $\chi$  and  $\psi$  having conductors  $f_\chi$  and  $f_\psi$ , respectively, we define the product,  $\chi\psi$ , to be the primitive character with  $\chi\psi(a) = \chi(a)\psi(a)$  for all  $a \in \mathbf{Z}$  such that  $(a, f_\chi f_\psi) = 1$ . Note that there may exist some values of  $a$  such that  $\chi\psi(a) \neq \chi(a)\psi(a)$ , due to the fact that our definition requires  $\chi\psi$  to be a primitive character. The conductor  $f_{\chi\psi}$  then divides  $\text{lcm}(f_\chi, f_\psi)$ . With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character  $\chi = 1$ , having conductor  $f_1 = 1$ . The inverse of the character  $\chi$  is the character  $\chi^{-1} = \overline{\chi}$ , the map of complex conjugates of the values of  $\chi$ .

Since any Dirichlet character  $\chi$  is multiplicative, we must have  $\chi(-1) = \pm 1$ . A character  $\chi$  is said to be odd if  $\chi(-1) = -1$ , and even if  $\chi(-1) = 1$ .

### 2.2 GENERALIZED BERNOULLI POLYNOMIALS

Let  $\chi$  be a Dirichlet character with conductor  $f_\chi$ . Then we define the functions,  $B_{n,\chi}(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by the generating function

$$(1) \quad \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with  $\chi$ ,  $B_{n,\chi}$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that  $B_{n,\chi}(0) = B_{n,\chi}$ . Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!},$$

and from this we obtain

$$(2) \quad B_{n,\chi}(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions  $B_{n,\chi}(t)$ , defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with  $\chi$ . Let  $\mathbf{Z}[\chi]$  denote the ring generated over  $\mathbf{Z}$  by all the values  $\chi(a)$ ,  $a \in \mathbf{Z}$ , and  $\mathbf{Q}(\chi)$  the field generated over  $\mathbf{Q}$  by all such values. Then it can be shown that  $f_\chi B_{n,\chi}$  must be in  $\mathbf{Z}[\chi]$  for each  $n \geq 0$  whenever  $\chi \neq 1$ . In general, we have  $B_{n,\chi} \in \mathbf{Q}(\chi)$  for each  $n \geq 0$ , and so  $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$ . The polynomials  $B_{n,\chi}(t)$  exhibit the property that, for all  $n \geq 0$ ,

$$(3) \quad B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever  $\chi \neq 1$ . Thus  $B_{n,\chi}(t)$ , for  $\chi \neq 1$ , is either an even function or an odd function according to whether  $(-1)^n \chi(-1)$  is 1 or  $-1$ . From (3) we obtain