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THEOREM 4.12. *Let  $n$ ,  $c$ , and  $k$  be positive integers, and let  $\tau \in \mathbf{Z}_p$  such that  $|\tau|_p \leq |pq^{-1}F_0|_p$ . Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

*and, modulo  $q\mathbf{Z}_p[\chi]$ , is independent of  $n$ .*

These results show that if related congruences hold for

$$\beta_{n,\chi}(0) = -\frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n,\chi_n},$$

then they must also hold for  $\beta_{n,\chi}(\tau)$ , where  $\tau$  is any element of  $\mathbf{Z}_p$  such that  $|\tau|_p \leq |pq^{-1}F_0|_p$ .

In [9] Granville defined ordinary Bernoulli numbers of negative index,  $B_{-n}$ , where  $n \in \mathbf{Z}$ ,  $n \geq 1$ , in the field  $\mathbf{Q}_p$  according to

$$B_{-n} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n},$$

where the limit is taken in the  $p$ -adic sense. In a similar manner we define generalized Bernoulli numbers of negative index,  $B_{-n,\chi}$ ,  $n \in \mathbf{Z}$ ,  $n \geq 1$ , and a collection of functions that correspond to generalized Bernoulli polynomials of negative index,  $B_{-n,\chi}(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 1$ . As a result of our definitions, we show that the  $B_{-n,\chi}(t)$  are actually power series that can be written in the form

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

converging for  $t \in \mathbf{C}_p$ ,  $|t|_p < 1$ . We close out by considering some properties of these functions.

## 2. PRELIMINARIES

The  $p$ -adic  $L$ -functions,  $L_p(s; \chi)$ , were first generated by Kubota and Leopoldt for the purpose of finding functions that would serve as analogues of the Dirichlet  $L$ -functions in the  $p$ -adic number field [14]. They are characterized by the fact that they interpolate a specific expression involving generalized Bernoulli numbers when the variable  $s$  is a nonpositive integer. In the following, for each  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , we derive a  $p$ -adic function  $L_p(s, \tau; \chi)$  that interpolates a specific expression involving generalized

Bernoulli polynomials in  $\tau$  for similar values of the variable  $s$ . These functions are designed so that  $L_p(s, 0; \chi) = L_p(s; \chi)$ . The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those  $\tau \in \overline{\mathbf{Q}}_p$  with  $|\tau|_p \leq 1$ . To complete the derivation we show that there exist functions  $L_p(s, \tau; \chi)$  for all  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , such that for every sequence  $\{\tau_i\}_{i=0}^\infty$  in  $\overline{\mathbf{Q}}_p$ , with  $|\tau_i|_p \leq 1$ , converging to some  $\tau \in \mathbf{C}_p$ , the sequence  $\{L_p(1-n, \tau_i; \chi)\}_{i=0}^\infty$ , with  $n \in \mathbf{Z}$ ,  $n \geq 1$ , converges to  $L_p(1-n, \tau; \chi)$ . Thus for each  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , the function  $L_p(s, \tau; \chi)$  must interpolate the appropriate expressions involving generalized Bernoulli polynomials for  $s = 1 - n$ ,  $n \in \mathbf{Z}$ ,  $n \geq 1$ .

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

## 2.1 DIRICHLET CHARACTERS

For  $n \in \mathbf{Z}$ ,  $n \geq 1$ , a Dirichlet character to the modulus  $n$  is a multiplicative map  $\chi : \mathbf{Z} \rightarrow \mathbf{C}$  such that  $\chi(a+n) = \chi(a)$  for all  $a \in \mathbf{Z}$ , and  $\chi(a) = 0$  if and only if  $(a, n) \neq 1$ . Since  $a^{\phi(n)} \equiv 1 \pmod{n}$  for all  $a$  such that  $(a, n) = 1$ ,  $\chi(a)$  must be a root of unity for such  $a$ .

If  $\chi$  is a Dirichlet character to the modulus  $n$ , then for any positive multiple  $m$  of  $n$  we can induce a Dirichlet character  $\psi$  to the modulus  $m$  according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a, m) = 1 \\ 0, & \text{if } (a, m) \neq 1. \end{cases}$$

The minimum modulus  $n$  for which a character  $\chi$  cannot be induced from some character to the modulus  $m$ ,  $m < n$ , is called the conductor of  $\chi$ , denoted  $f_\chi$ . We shall assume that each  $\chi$  is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters  $\chi$  and  $\psi$  having conductors  $f_\chi$  and  $f_\psi$ , respectively, we define the product,  $\chi\psi$ , to be the primitive character with  $\chi\psi(a) = \chi(a)\psi(a)$  for all  $a \in \mathbf{Z}$  such that  $(a, f_\chi f_\psi) = 1$ . Note that there may exist some values of  $a$  such that  $\chi\psi(a) \neq \chi(a)\psi(a)$ , due to the fact that our definition requires  $\chi\psi$  to be a primitive character. The conductor  $f_{\chi\psi}$  then divides  $\text{lcm}(f_\chi, f_\psi)$ . With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character  $\chi = 1$ , having conductor  $f_1 = 1$ . The inverse of the character  $\chi$  is the character  $\chi^{-1} = \overline{\chi}$ , the map of complex conjugates of the values of  $\chi$ .

Since any Dirichlet character  $\chi$  is multiplicative, we must have  $\chi(-1) = \pm 1$ . A character  $\chi$  is said to be odd if  $\chi(-1) = -1$ , and even if  $\chi(-1) = 1$ .

## 2.2 GENERALIZED BERNOULLI POLYNOMIALS

Let  $\chi$  be a Dirichlet character with conductor  $f_\chi$ . Then we define the functions,  $B_{n,\chi}(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by the generating function

$$(1) \quad \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with  $\chi$ ,  $B_{n,\chi}$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that  $B_{n,\chi}(0) = B_{n,\chi}$ . Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!},$$

and from this we obtain

$$(2) \quad B_{n,\chi}(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions  $B_{n,\chi}(t)$ , defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with  $\chi$ . Let  $\mathbf{Z}[\chi]$  denote the ring generated over  $\mathbf{Z}$  by all the values  $\chi(a)$ ,  $a \in \mathbf{Z}$ , and  $\mathbf{Q}(\chi)$  the field generated over  $\mathbf{Q}$  by all such values. Then it can be shown that  $f_\chi B_{n,\chi}$  must be in  $\mathbf{Z}[\chi]$  for each  $n \geq 0$  whenever  $\chi \neq 1$ . In general, we have  $B_{n,\chi} \in \mathbf{Q}(\chi)$  for each  $n \geq 0$ , and so  $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$ . The polynomials  $B_{n,\chi}(t)$  exhibit the property that, for all  $n \geq 0$ ,

$$(3) \quad B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever  $\chi \neq 1$ . Thus  $B_{n,\chi}(t)$ , for  $\chi \neq 1$ , is either an even function or an odd function according to whether  $(-1)^n \chi(-1)$  is 1 or  $-1$ . From (3) we obtain



$$B_{n,\chi} = (-1)^n \chi(-1) B_{n,\chi},$$

and so  $B_{n,\chi} = 0$  whenever  $n$  is even and  $\chi$  is odd, or whenever  $n$  is odd and  $\chi$  is even,  $\chi \neq 1$ . Another property that the polynomials satisfy is that for  $m \in \mathbf{Z}$ ,  $m \geq 1$ ,

$$(4) \quad B_{n,\chi}(mf_\chi + t) - B_{n,\chi}(t) = n \sum_{a=1}^{mf_\chi} \chi(a)(a+t)^{n-1},$$

for all  $n \geq 0$ . This can be derived from (1). Note that for  $\chi = 1$  and  $t = 0$  this becomes

$$\frac{1}{n} (B_{n,1}(m) - B_{n,1}) = \sum_{a=1}^m a^{n-1}.$$

If  $\chi \neq 1$ , then it can be shown that  $\sum_{a=1}^{f_\chi} \chi(a) = 0$ , and from the above relations we can derive

$$B_{0,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a)$$

for all  $\chi$ . Therefore

$$B_{0,\chi} = \begin{cases} 0, & \text{if } \chi \neq 1 \\ 1, & \text{if } \chi = 1. \end{cases}$$

The ordinary Bernoulli polynomials,  $B_n(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , are defined by

$$(5) \quad \frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and the Bernoulli numbers,  $B_n$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi.$$

From this we obtain the values  $B_0 = 1$ ,  $B_1 = -1/2$ ,  $B_2 = 1/6$ ,  $B_4 = -1/30$ , ..., with  $B_n = 0$  for odd  $n \geq 3$ . For even  $n \geq 2$ , we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m.$$

Note that we again have the relations  $B_n(0) = B_n$  and

$$B_n(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m} t^m,$$

as we did for the generalized Bernoulli polynomials.

Some of the more important properties of Bernoulli polynomials are that

$$(6) \quad B_n(t+1) - B_n(t) = nt^{n-1}$$

for all  $n \geq 1$ , and

$$B_n(1-t) = (-1)^n B_n(t)$$

for  $n \geq 0$ . Each of these results can be derived from the generating function (5) above.

Similar to (4) for the generalized Bernoulli polynomials, whenever  $m, n \in \mathbf{Z}$ ,  $m \geq 1$ ,  $n \geq 1$ ,

$$\frac{1}{n} (B_n(m) - B_n) = \sum_{a=0}^{m-1} a^{n-1},$$

where we take  $0^0$  to be 1 in the case of  $a = 0$  and  $n = 1$ . Note that this can be derived from (6) since

$$B_n(m) - B_n = \sum_{a=0}^{m-1} (B_n(a+1) - B_n(a)).$$

The Bernoulli numbers are rational numbers, and, in fact, the von Staudt-Clausen theorem states that for even  $n \geq 2$ ,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus the denominator of each  $B_n$  must be square-free.

The ordinary Bernoulli numbers are related to the generalized Bernoulli numbers in that for  $\chi = 1$  we have

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n,1} \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and since

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1},$$

we see that  $B_{n,1} = B_n$  for all  $n \neq 1$ , and  $B_{1,1} = -B_1$ . In fact, this can be written as  $B_{n,1} = (-1)^n B_n$ , and for the polynomials,  $B_{n,1}(t) = (-1)^n B_n(-t)$ .

### 2.3 DIRICHLET $L$ -FUNCTIONS

For  $\chi$  a Dirichlet character with conductor  $f_\chi$ , the Dirichlet  $L$ -function for  $\chi$  is defined by

$$L(s; \chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

for  $s \in \mathbf{C}$  such that  $\Re(s) > 1$ . Note that  $L(s; \chi)$  can be continued analytically to all of  $\mathbf{C}$ , except for a pole of order 1 at  $s = 1$  when  $\chi = 1$ .

Let  $\tau(\chi)$  be a Gauss sum,

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a / f_\chi},$$

where  $i^2 = -1$ , and let

$$\delta_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

Then  $L(s; \chi)$  satisfies the functional equation

$$(7) \quad \left(\frac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\frac{s + \delta_\chi}{2}\right) L(s; \chi) = W_\chi \left(\frac{f_\chi}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s + \delta_\chi}{2}\right) L(1-s; \bar{\chi}),$$

where  $\Gamma(s)$  is the gamma function, and  $W_\chi = \frac{\tau(\chi)}{i^{\delta_\chi} \sqrt{f_\chi}}$ , having the property that  $|W_\chi| = 1$ . Since  $\Gamma(s)$  has simple poles at the negative integers,  $L(s; \chi)$  must be zero for  $s = 1 - n$ , where  $n \in \mathbf{Z}$ ,  $n \geq 1$ , such that  $n \not\equiv \delta_\chi \pmod{2}$ , except when  $\chi = 1$  and  $n = 1$ .  $L(s; \chi)$  can also be described by means of the Euler product  $L(s; \chi) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}$ , for  $s \in \mathbf{C}$  such that  $\Re(s) > 1$ . Thus  $L(s; \chi) \neq 0$  in this domain.

The generalized Bernoulli numbers,  $B_{n, \chi}$ , and the Dirichlet  $L$ -function,  $L(s; \chi)$ , share the following relationship, a proof of this being found in [13]:

**THEOREM 2.1.** *Let  $\chi$  be a Dirichlet character, and let  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Then  $L(1 - n; \chi) = -\frac{1}{n} B_{n, \chi}$ .*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.

## 2.4 THE $p$ -ADIC NUMBER FIELD

Let  $p$  be prime. We shall use  $\mathbf{Z}_p$  to represent the  $p$ -adic integers, and  $\mathbf{Q}_p$  the  $p$ -adic rationals. Let  $|\cdot|_p$  denote the  $p$ -adic absolute value on  $\mathbf{Q}_p$ , normalized so that  $|p|_p = p^{-1}$ . Let  $\overline{\mathbf{Q}}_p$  be the algebraic closure of  $\mathbf{Q}_p$ . The absolute value on  $\mathbf{Q}_p$  extends uniquely to  $\overline{\mathbf{Q}}_p$ , however  $\overline{\mathbf{Q}}_p$  is not complete with respect to the absolute value. Let  $\mathbf{C}_p$  be the completion of  $\overline{\mathbf{Q}}_p$  with respect to this absolute value. Then the absolute value extends to  $\mathbf{C}_p$ , and  $\overline{\mathbf{Q}}_p$  is dense in  $\mathbf{C}_p$ . We also have  $\mathbf{C}_p$  algebraically closed. Furthermore, on  $\mathbf{C}_p$  the absolute value is non-Archimedean, and so

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

for any  $a, b \in \mathbf{C}_p$ . Note that the two fields  $\mathbf{C}$  and  $\mathbf{C}_p$  are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of  $\mathbf{C}_p$  in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \quad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then  $\mathfrak{p}$  is a maximal ideal of  $\mathfrak{o}$ . If  $\tau \in \mathbf{C}_p$  such that  $|\tau|_p \leq |p|_p^s$ , where  $s \in \mathbf{Q}$ , then  $\tau \in p^s \mathfrak{o}$ , and so we shall also write this as  $\tau \equiv 0 \pmod{p^s \mathfrak{o}}$ .

Any  $n \in \mathbf{Z}$ ,  $n > 0$ , can be uniquely expressed in the form  $n = \sum_{m=0}^k a_m p^m$ , where  $a_m \in \mathbf{Z}$ ,  $0 \leq a_m \leq p-1$ , for  $m = 0, 1, \dots, k$ , and  $a_k \neq 0$ . For such  $n$ , we define

$$s_p(n) = \sum_{m=0}^k a_m,$$

the sum of the  $p$ -adic digits of  $n$ , and also define  $s_p(0) = 0$ . For any  $n \in \mathbf{Z}$ , let  $v_p(n)$  be the highest power of  $p$  dividing  $n$ . This function is additive, and relates to the function  $s_p(n)$  by means of the identity

$$(8) \quad v_p(n!) = \frac{n - s_p(n)}{p-1},$$

which holds for all  $n \geq 0$ . Note that for  $n \geq 1$  this implies that

$$v_p(n!) \leq \frac{n-1}{p-1}.$$

The definition of this function can be extended to all of  $\mathbf{Q}$  by taking  $v_p(1/n) = -v_p(n)$ .

Throughout we let  $q = 4$  if  $p = 2$ , and  $q = p$  otherwise. Note that there exist  $\phi(q)$  distinct solutions, modulo  $q$ , to the equation  $x^{\phi(q)} - 1 = 0$ , and each solution must be congruent to one of the values  $a \in \mathbf{Z}$ , where  $1 \leq a \leq q$ ,

$(a, p) = 1$ . Thus, by Hensel's Lemma, given  $a \in \mathbf{Z}$  with  $(a, p) = 1$ , there exists a unique  $\omega(a) \in \mathbf{Z}_p$ , where  $\omega(a)^{\phi(q)} = 1$ , such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting  $\omega(a) = 0$  for  $a \in \mathbf{Z}$  such that  $(a, p) \neq 1$ , we see that  $\omega$  is actually a Dirichlet character, called the Teichmüller character, having conductor  $f_\omega = q$ . Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then  $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$ . For  $p \geq 3$ ,  $\lim_{n \rightarrow \infty} a^{p^n} = \omega(a)$ , since  $a^{p^n} \equiv a \pmod{p}$  and  $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$ .

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character  $\omega$ . If  $t \in \mathbf{C}_p$  such that  $|t|_p \leq 1$ , then for any  $a \in \mathbf{Z}$ ,  $a + qt \equiv a \pmod{q\mathfrak{o}}$ . Thus we define

$$\omega(a + qt) = \omega(a)$$

for these values of  $t$ . We also define

$$\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$$

for such  $t$ .

Fix an embedding of the algebraic closure of  $\mathbf{Q}$ ,  $\overline{\mathbf{Q}}$ , into  $\mathbf{C}_p$ . We may then consider the values of a Dirichlet character  $\chi$  as lying in  $\mathbf{C}_p$ . For  $n \in \mathbf{Z}$  we define the product  $\chi_n = \chi\omega^{-n}$  in the sense of the product of characters. This implies that  $f_{\chi_n} \mid f_\chi q$ . However, since we can write  $\chi = \chi_n\omega^n$ , we also have  $f_\chi \mid f_{\chi_n} q$ . Thus  $f_\chi$  and  $f_{\chi_n}$  differ by a factor that is a power of  $p$ . In fact, either  $f_{\chi_n}/f_\chi \in \mathbf{Z}$  and divides  $q$ , or  $f_\chi/f_{\chi_n} \in \mathbf{Z}$  and divides  $q$ .

Let  $\mathbf{Q}_p(\chi)$  denote the field generated over  $\mathbf{Q}_p$  by all values  $\chi(a)$ ,  $a \in \mathbf{Z}$ . In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field  $\mathbf{Q}_p(\chi)$ , for all  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} (B_{n+1,\chi}(p^h f_\chi) - B_{n+1,\chi}(0)).$$

From this we can obtain

LEMMA 2.3. Let  $\tau \in \mathbf{C}_p$ . In the field  $\mathbf{Q}_p(\chi, \tau)$ , for all  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a + \tau)^n.$$

*Proof.* By applying Lemma 2.2 to (4), we obtain

$$B_{n, \chi} = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi(a)a^n.$$

Therefore, by (2),

$$\begin{aligned} B_{n, \chi_n}(\tau) &= \sum_{m=0}^n \binom{n}{m} \tau^{n-m} \lim_{h \rightarrow \infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h \rightarrow \infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) \sum_{m=0}^n \binom{n}{m} \tau^{n-m} a^m. \end{aligned}$$

Since  $f_\chi$  and  $f_{\chi_n}$  differ by a factor that is a power of  $p$ , we must have

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a + \tau)^n,$$

and the proof is complete.  $\square$

## 2.5 $p$ -ADIC FUNCTIONS

Let  $K$  be an extension of  $\mathbf{Q}_p$  contained in  $\mathbf{C}_p$ . An infinite series  $\sum_{n=0}^{\infty} a_n$ ,  $a_n \in K$ , converges in  $K$  if and only if  $|a_n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $K[[x]]$  be the algebra of formal power series in  $x$ . Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in  $K[[x]]$ , converges at  $x = \xi$ ,  $\xi \in \mathbf{C}_p$ , if and only if  $|a_n \xi^n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore whenever a power series  $A(x)$  converges at some  $\xi_0 \in \mathbf{C}_p$ , then it must converge at all  $\xi \in \mathbf{C}_p$  such that  $|\xi|_p \leq |\xi_0|_p$ . The following result, for double series in  $K$ , can be found in [8].

PROPOSITION 2.4. Let  $b_{n,m} \in K$ , and suppose that for each  $\epsilon > 0$  there exists  $N \in \mathbb{Z}$ , depending on  $\epsilon$ , such that if  $\max\{n, m\} \geq N$ , then  $|b_{n,m}|_p \leq \epsilon$ . Then both series

$$\sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} b_{n,m} \right) \quad \text{and} \quad \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} b_{n,m} \right)$$

converge, and their sums are equal.

There are two power series that we wish to make note of in particular. First we define the  $p$ -adic exponential function,  $\exp(x)$ , in  $\mathbb{Q}_p[[x]]$ , by

$$(9) \quad \exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

From (8) we can conclude that this power series converges in  $\{x \in \mathbb{C}_p : |x|_p < p^{-1/(p-1)}\}$ . The  $p$ -adic logarithm function,  $\log(x)$ , in  $\mathbb{Q}_p[[x]]$ , is defined by

$$(10) \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n,$$

the power series converging in the domain  $\{x \in \mathbb{C}_p : |x|_p < 1\}$ . For  $|x|_p < p^{-1/(p-1)}$ , we have  $\log(\exp(x)) = x$  and  $\exp(\log(1+x)) = 1+x$ .

The following property is a uniqueness property for power series, found in [13].

LEMMA 2.5. Let  $A(x), B(x) \in K[[x]]$ , such that each converges in a neighborhood of 0 in  $\mathbb{C}_p$ . If  $A(\xi_n) = B(\xi_n)$  for a sequence  $\{\xi_n\}_{n=0}^{\infty}$ ,  $\xi_n \neq 0$ , in  $\mathbb{C}_p$ , such that  $\xi_n \rightarrow 0$ , then  $A(x) = B(x)$ .

Let  $U$  be an open subset of  $\mathbb{C}_p$ , contained in the domain of the  $p$ -adic function  $f$ . We say that  $f$  is differentiable at  $x \in U$  if the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists. If this limit exists for each  $x \in U$ , then we say that  $f$  is differentiable in  $U$ .

The relationship between the derivatives of a function and its power series expansion is given in the following result, found in [8].

PROPOSITION 2.6. Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with coefficients in  $\mathbf{C}_p$ , and suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - \alpha)^n$$

converges on some closed ball  $B$  in  $\mathbf{C}_p$ . Then

i) For each  $x \in B$ , the  $k^{\text{th}}$  derivative  $f^{(k)}(x)$  exists, and is given by

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x - \alpha)^{n-k},$$

and we have

$$a_k = \frac{1}{k!} f^{(k)}(\alpha).$$

ii) Let  $\beta \in B$ . Then there exists a series  $\sum_{n=0}^{\infty} b_n x^n$  such that

$$f(x) = \sum_{n=0}^{\infty} b_n (x - \beta)^n$$

for any  $x \in B$ . Both series  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=0}^{\infty} b_n x^n$  have the same region of convergence.

Now let  $K$  be a finite extension of  $\mathbf{Q}_p$ . For  $A(x) \in K[[x]]$ ,  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ , where  $a_n \in K$ , define

$$\|A\| = \sup_n |a_n|_p.$$

Let  $P_K = \{A(x) \in K[[x]] : \|A\| < \infty\}$ . Then  $\|\cdot\|$  defines a norm on  $P_K$ , and so  $K[x] \subset P_K \subset K[[x]]$ . Furthermore  $P_K$  is complete in this norm.

Let  $\{b_n\}_{n=0}^{\infty}$  be a sequence of elements of  $K$ , and let the sequence  $\{c_n\}_{n=0}^{\infty}$  be defined by

$$(11) \quad c_n = \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} b_m$$

for each  $n \in \mathbf{Z}$ ,  $n \geq 0$ . Then  $c_n \in K$  for each  $n \geq 0$ . Note that (11) implies that these sequences must satisfy

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = e^{-t} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$



This implies that

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and so we have the relationship

$$(12) \quad b_n = \sum_{m=0}^n \binom{n}{m} c_m$$

for each  $n \in \mathbf{Z}$ ,  $n \geq 0$ . We can reverse this process to derive (11) given (12). Thus (11) and (12) must be equivalent. The following relate to sequences that satisfy (11) and (12), and are found in [13].

**THEOREM 2.7.** *Let  $\{b_n\}_{n=0}^{\infty}$  and  $\{c_n\}_{n=0}^{\infty}$  be defined as in the above relation. Let  $\rho \in \mathbf{R}$  such that  $0 < \rho < |p|_p^{1/(p-1)}$ . If  $|c_n|_p \leq C\rho^n$  for all  $n \geq 0$ , where  $C > 0$ , then there exists a unique power series  $A(x) \in P_K$  such that  $A(x)$  converges at every  $\xi \in \mathbf{C}_p$  with  $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$ , and  $A(n) = b_n$  for every  $n \geq 0$ .*

**COROLLARY 2.8.** *Let  $A(x)$  be the power series from the theorem. Then for each  $\xi \in \mathbf{C}_p$  such that  $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$ , we have*

$$A(\xi) = \sum_{n=0}^{\infty} c_n \binom{\xi}{n}.$$

Theorem 2.7 can be applied to the sequence  $\{b_n\}_{n=0}^{\infty}$  in  $K = \mathbf{Q}_p(\chi)$ , where

$$b_n = (1 - \chi_n(p)p^{n-1}) B_{n,\chi_n},$$

in order to obtain a power series  $A_\chi(s)$  satisfying  $A_\chi(n) = b_n$ , and converging on the domain  $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$ . (Since  $|p|_p^{1/(p-1)}|q|_p^{-1} > 1$  and  $|n|_p \leq 1$  for each  $n \in \mathbf{Z}$ , all of  $\mathbf{Z}$  is contained in this domain.) From this a  $p$ -adic function,  $L_p(s; \chi)$ , can be derived that interpolates the values

$$L_p(1 - n; \chi) = -\frac{1}{n} b_n,$$

and which converges in  $\{s \in \mathbf{C}_p : |s - 1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$ , except  $s \neq 1$  if  $\chi = 1$ . Note that if  $\chi$  is odd, then  $\chi_n$  is even when  $n$  is odd, and  $\chi_n$  is odd when  $n$  is even. Thus the quantity  $(1 - \chi_n(p)p^{n-1})B_{n,\chi_n} = 0$  for all  $n \in \mathbf{Z}$ ,  $n \geq 1$ , as we saw from the properties of generalized Bernoulli numbers. Therefore  $L_p(s; \chi)$  vanishes on a sequence such as  $\{-p^m\}_{m=0}^{\infty}$ , which has 0 as a limit point, implying that for such  $\chi$  we must have  $L_p(s; \chi) \equiv 0$ .