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A p-ADIC L-FUNCTION OF TWO VARIABLES

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ABSTRACT. For p prime and χ a primitive Dirichlet character, we derive a p-adic function $L_p(s,t;\chi)$, where $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathbf{C}_p$, $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, $s \neq 1$ if $\chi = 1$, with $q = 4$ if $p = 2$ and $q = p$ if $p > 2$, that interpolates the values

$$L_p(1-n,t;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}(p^{-1}qt)\right),$$

for $n \in \mathbf{Z}$, $n \geq 1$. Here $B_{n,\chi}(t)$ is the n^{th} generalized Bernoulli polynomial associated with the character χ, and $\chi_n = \chi\omega^{-n}$, where ω is the Teichmüller character. This function is then a two-variable analogue of the p-adic L-function $L_p(s;\chi)$, where $s \in \mathbf{C}_p$, $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, $s \neq 1$ if $\chi = 1$, in that this function satisfies $L_p(s,0;\chi) = L_p(s;\chi)$. In addition to deriving this function, we establish several properties and applications of $L_p(s,t;\chi)$.

1. INTRODUCTION

Given a primitive Dirichlet character χ, having conductor f_χ (see Section 2 for definitions), the Dirichlet L-function associated with χ is defined by

$$L(s;\chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

where $s \in \mathbf{C}$, $\Re(s) > 1$. This function can be continued analytically to the entire complex plane, except for a simple pole at $s = 1$ when $\chi = 1$, in which case we have the Riemann zeta function, $\zeta(s) = L(s;1)$. It is believed that the analysis of Dirichlet L-functions began with Euler's study of $\zeta(s)$, in which he considered the function only for real values of s. It was Riemann

*) A majority of these results were obtained while the author was a graduate student at the University of Georgia, Athens, under the direction of Andrew Granville.

who extended this study to a complex variable [17]. Of notable interest are the values of $L(s;\chi)$ at $s=n$, $n\in\mathbf{Z}$. Euler was able to evaluate $\zeta(s)$ at the positive even integers. However, the determination of the values of this function at odd $s\geq 3$ remains an open problem. Similarly, the values of $L(s;\chi)$ can be determined at either the positive even or odd integers depending on the sign of $\chi(-1)$. Furthermore, these functions can be readily evaluated at all integer values of $s\leq 0$. Because of a functional equation (7) that the Dirichlet L-functions satisfy (discovered by Riemann [17] for $\zeta(s)$), we can obtain a relationship between the values of $L(s;\chi)$ at positive and negative $s\in\mathbf{Z}$.

Jakob Bernoulli was the first to consider a particular sequence of rational numbers in the study of finite sums of a given power of consecutive integers [4]. In this study, he gave a defining relationship that enables the generation of this sequence. This sequence of numbers has, since that time, come to be known as the Bernoulli numbers, B_n, $n\in\mathbf{Z}$, $n\geq 0$. They are given by $B_0=1, B_1=-1/2, B_2=1/6, B_3=0, B_4=-1/30,\ldots$, where $B_n=0$ for odd $n\geq 3$, and for all $n\geq 1$,

$$B_n=-\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m.$$

The Bernoulli polynomials were first introduced by Raabe in [16]. They can be expressed in the form

$$B_n(t)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}t^m,$$

where $n\in\mathbf{Z}$, $n\geq 0$. The form in which they are currently defined has been somewhat modified from Raabe's original construction, but the results that he obtained set the framework for a continuing history of analysis on these polynomials.

The generalized Bernoulli numbers associated with the Dirichlet character χ, $B_{n,\chi}$, $n\in\mathbf{Z}$, $n\geq 0$, were defined in [12], [3], [1], and [15]. We obtain the standard Bernoulli numbers when $\chi=1$, in that $B_{n,1}=B_n$ if $n\neq 1$, and $B_{1,1}=-B_1$. The generalized Bernoulli numbers share a particular relationship with the Dirichlet L-function, $L(s;\chi)$, in that

$$L(1-n;\chi)=-\frac{1}{n}B_{n,\chi},$$

for $n\in\mathbf{Z}$, $n\geq 1$. The generalized Bernoulli polynomials, $B_{n,\chi}(t)$, are given by

$$B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi} t^m,$$

where $n \in \mathbf{Z}$, $n \geq 0$.

During the development of p-adic analysis, effort was made to derive a meromorphic function, defined over the p-adic number field, that would interpolate the same, or at least similar, values as the Dirichlet L-function at nonpositive integers. In [14] Kubota and Leopoldt proved the existence of such a function, considered the p-adic equivalent of the Dirichlet L-function. This function, $L_p(s;\chi)$, yields the values

$$L_p(1-n;\chi) = -\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

for $n \in \mathbf{Z}$, $n \geq 1$, where $\chi_n = \chi\omega^{-n}$, with ω the Teichmüller character. The function $L_p(s;\chi)$ can be expressed in the form

$$L_p(s;\chi) = \frac{a_{-1}}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n,$$

where

$$a_{-1} = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1, \end{cases}$$

and $a_n \in \mathbf{Q}_p(\chi)$, a finite extension of $\mathbf{Q}_p$, for $n \geq 0$. The power series given in the above expression converges in $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < r\}$, for $r = |p|_p^{1/(p-1)}|q|_p^{-1}$, where $q = 4$ if $p = 2$, and $q = p$ otherwise. Much additional information about these functions can be found in [19].

We have found a more general form for the p-adic L-function $L_p(s;\chi)$. Instead of generating a function of one variable that interpolates an expression involving generalized Bernoulli numbers, we have sought out a function of two variables that in one variable interpolates an expression that involves generalized Bernoulli polynomials in the other variable, such that when this second variable is 0, we obtain the familiar function $L_p(s;\chi)$. We have constructed such a function for all primes p, and so we have been able to prove the existence of a p-adic L-function, $L_p(s,t;\chi)$, where $s \in \mathbf{C}_p$ such that $|s-1|_p < r$, except $s \neq 1$ when $\chi = 1$, and $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, which interpolates the polynomials

$$L_p(1-n,t;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

for $n \in \mathbf{Z}$, $n \geq 1$. This function also has an expansion

$$L_p(s,t;\chi) = \frac{a_{-1}(t)}{s-1} + \sum_{n=0}^{\infty} a_n(t)(s-1)^n,$$

where

$$a_{-1}(t) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1\,. \end{cases}$$

If $\chi(-1) = -1$, then $B_{n,\chi_n} = 0$ for each $n \geq 0$. Thus the corresponding p-adic L-function, $L_p(s;\chi)$, vanishes on a set that has a limit point in $\mathbf{Z}_p$. This implies that $L_p(s;\chi)$ must vanish identically for all $s \in \mathfrak{D}$. Because of this, proofs of the existence of this function need only deal with the case of those χ such that $\chi(-1) = 1$, and properties associated with these χ can then be utilized to enhance the efficiency of the proof. In the more generalized form, the p-adic L-function $L_p(s,t;\chi)$ must satisfy $L_p(s,0;\chi) = L_p(s;\chi)$, and so $L_p(s,0;\chi)$ vanishes for all $s \in \mathfrak{D}$ when $\chi(-1) = -1$, but this property does not hold for all t for any given χ. Thus we cannot focus the proof of the existence of $L_p(s,t;\chi)$ solely on those χ such that $\chi(-1) = 1$.

In Section 3, we derive $L_p(s,t;\chi)$ according to the method given in [13], Chapter 3. In this method, if a sequence $\{b_n\}_{n=0}^{\infty}$, in a finite extension of $\mathbf{Q}_p$, is given such that

$$c_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m$$

satisfies $|c_n|_p \leq C\rho^n$, for all $n \geq 0$, where $C, \rho \in \mathbf{R}$, with $C > 0$ and $0 < \rho < |p|_p^{1/(p-1)}$, then a power series $A(s)$ can be generated such that $A(n) = b_n$, for each n, and such that $A(s)$ converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}\rho^{-1}\}$. It is then shown that, given a Dirichlet character χ, the values $b_n = (1 - \chi_n(p)p^{n-1})B_{n,\chi_n}$, $n \geq 0$, form such a sequence, and thus we have a power series $A_\chi(s)$ which interpolates the b_n and which converges in the domain $\mathfrak{D}$. The p-adic L-function, $L_p(s;\chi)$, is generated by taking $L_p(s;\chi) = (s-1)^{-1}A_\chi(1-s)$.

In our work we first let τ be an element of a finite field extension of $\mathbf{Q}_p$, contained in the algebraic closure, $\overline{\mathbf{Q}}_p$, of $\mathbf{Q}_p$, with $|\tau|_p \leq 1$. We then define the sequence $\{b_n(\tau)\}_{n=0}^{\infty}$ by

$$b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right).$$

The sequence $\{c_n(\tau)\}_{n=0}^{\infty}$ is defined as above, and we prove

PROPOSITION 3.3. *For all* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and for* $n \in \mathbf{Z}$, $n \geq 0$, *we have* $|c_n(\tau)|_p \leq |pqf_\chi|_p^{-1}|q|_p^n$.

At this point it follows that a p-adic power series $A_\chi(s,\tau)$ exists, satisfying $A_\chi(n,\tau) = b_n(\tau)$, and converging in $\mathfrak{D}$. We can then form the

p-adic function $L_p(s,\tau;\chi)$, satisfying $L_p(1-n,\tau;\chi) = -b_n(\tau)/n$, by taking $L_p(s,\tau;\chi) = (s-1)^{-1}A_\chi(1-s,\tau)$. However, this is only for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. In order to prove this for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we derive a means of defining $L_p(s,\tau;\chi)$ for each such τ, and then prove the following:

LEMMA 3.12. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Then for each $n \in \mathbf{Z}$, $n \geq 1$,*

$$\lim_{i\to\infty} L_p(1-n,\tau_i;\chi) = L_p(1-n,\tau;\chi).$$

Therefore, as a consequence of this, we deduce

THEOREM 3.13. *For each $\tau \in \mathbf{C}_p$, with $|\tau|_p \leq 1$, there exists a unique p-adic, meromorphic function $L_p(s,\tau;\chi)$ that satisfies*

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, this function can be expressed in the form

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, and

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

Once we have established the existence of $L_p(s,\tau;\chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we proceed to investigate the properties of the two variable function $L_p(s,t;\chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$. In Section 4 we derive the following for all primes p:

THEOREM 4.3. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then $L_p(s,-t;\chi) = \chi(-1)L_p(s,t;\chi)$.*

This property follows from a similar property for the generalized Bernoulli polynomials. An immediate consequence of this is that $L_p(s;\chi) = 0$ when χ is odd. Another property of $L_p(s,t;\chi)$ is given by

LEMMA 4.6. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$\frac{\partial^n}{\partial t^n} L_p(s,t;\chi) = n!q^n \binom{-s}{n} L_p(s+n,t;\chi_n),$$

for $n \in \mathbf{Z}$, $n \geq 0$.

Here we are taking

$$\binom{-s}{n} L_p(s+n,t;\chi)\bigg|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi},$$

for $n \in \mathbf{Z}$, $n \geq 1$. Note that this result implies that

$$\frac{\partial^{p-1}}{\partial t^{p-1}} L_p(s,t;\chi) = (p-1)!q^{p-1} \binom{-s}{p-1} L_p(s+p-1,t;\chi).$$

Because of this lemma we can find a power series expansion of $L_p(s,t;\chi)$ in the variable t about any $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m).$$

When $\alpha = 0$, this theorem yields an expansion of $L_p(s,t;\chi)$ in terms of $L_p(s;\chi_m)$ for $m \in \mathbf{Z}$, and thus yields an additional method of derivation of $L_p(s,t;\chi)$.

Let $F_0 = \operatorname{lcm}(f_\chi, q)$, and let F be a positive multiple of $pq^{-1}F_0$. If we define $\langle a+qt \rangle = \omega^{-1}(a)(a+qt)$ for $a \in \mathbf{Z}$, $(a,p)=1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where ω is the Teichmüller character, then we have the following:

THEOREM 4.8. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s,t+F;\chi) - L_p(s,t;\chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt \rangle^{-s}.$$

We then have a connection between certain finite sums and the function $L_p(s,t;\chi)$. As a result of this, we obtain

COROLLARY 4.9. *Let* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a \rangle^{-s}.$$

Thus, when t takes on certain values, we have a finite expression for $L_p(s, t; \chi)$ in terms of previously known functions.

By combining the previous two theorems, we can obtain the relation

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a \rangle^{-s} = -\sum_{m=1}^{\infty} \binom{-s}{m} q^m F^m L_p(s+m; \chi_m),$$

where F is a positive multiple of $pq^{-1}F_0$, $F_0 = \mathrm{lcm}(f_\chi, q)$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. This is a generalization of a result of Barsky found in [2] (see also [20]).

A number of congruences relating to the ordinary and the generalized Bernoulli numbers have found a considerable amount of interest. One of the more notable examples is the Kummer congruence for the ordinary Bernoulli numbers, which states that $p^{-1}\Delta_c \frac{1}{n} B_n \in \mathbf{Z}_p$, where $c \in \mathbf{Z}$ is positive with $c \equiv 0 \pmod{p-1}$, and $n \in \mathbf{Z}$ is positive, even, and $n \not\equiv 0 \pmod{p-1}$ (see [19], p. 61). Note that we are using Δ_c to denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$, so that

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

More generally, it can be shown that $p^{-k}\Delta_c^k \frac{1}{n} B_n \in \mathbf{Z}_p$, where $k \in \mathbf{Z}$, with $k \geq 1$, and c and n are as above, but with $n > k$.

The application of Kummer's congruence to generalized Bernoulli numbers was first treated by Carlitz in [5], with the result that $p^{-k}\Delta_c^k \frac{1}{n} B_{n,\chi} \in \mathbf{Z}_p[\chi]$, for positive $c \in \mathbf{Z}$ with $c \equiv 0 \pmod{p-1}$, $n, k \in \mathbf{Z}$ with $n > k \geq 1$, and χ such that $f_\chi \neq p^\mu$, where $\mu \in \mathbf{Z}$, $\mu \geq 0$. From [7] (see also [18]) we see that if the operator Δ_c^k is applied to the quantity $-(1 - \chi_n(p)p^{n-1})B_{n,\chi_n}/n$, the value of $L_p(1-n; \chi)$, for similar c and characters χ, then the congruence will still hold if the restriction $n > k$ is dropped, requiring only that $n \geq 1$. In addition to this, the divisibility requirements on c can be removed, yielding a congruence of the form

$$q^{-k}\Delta_c^k \frac{1}{n} \left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n} \in \mathbf{Z}_p[\chi],$$

for $c, n, k \in \mathbf{Z}$, each positive, and χ such that $f_\chi \neq p^\mu$, $\mu \in \mathbf{Z}$, $\mu \geq 0$. Recall that we are taking $q = 4$ if $p = 2$, and $q = p$ otherwise. If we denote

$$\beta_{n,\chi} = -\frac{1}{n}\left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

then this congruence can be expressed as $q^{-k}\Delta_c^k \beta_{n,\chi} \in \mathbf{Z}_p[\chi]$.

As an extension of the Kummer congruence, Gunaratne (see [10], [11]) has shown that if $p > 3$, $c, n, k \in \mathbf{Z}$ are positive, and $\chi = \omega^h$, where $h \in \mathbf{Z}$ and $h \not\equiv 0 \pmod{p-1}$, then the value of $p^{-k}\Delta_c^k \beta_{n,\chi}$ modulo $p\mathbf{Z}_p$ is independent of n, and further satisfies

$$p^{-k}\Delta_c^k \beta_{n,\chi} \equiv p^{-k'}\Delta_c^{k'} \beta_{n',\chi} \pmod{p\mathbf{Z}_p}$$

for positive $n', k' \in \mathbf{Z}$ with $k \equiv k' \pmod{p-1}$. Additionally, by means of the binomial coefficient operator

$$\binom{p^{-1}\Delta_c}{k} x_n = \frac{1}{k!}\left(\prod_{j=0}^{k-1}(p^{-1}\Delta_c - j)\right) x_n,$$

for these χ we have $\binom{p^{-1}\Delta_c}{k}\beta_{n,\chi} \in \mathbf{Z}_p$, with a value modulo $p\mathbf{Z}_p$ that is independent of n.

By utilizing Corollary 4.9, we can derive a collection of congruences, similar to the results of Gunaratne, relating to the generalized Bernoulli polynomials, but without a restriction on either p or χ.

THEOREM 4.10. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq \left|pq^{-1}F_0\right|_p$. Then the quantity $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.*

Here we denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

the value of $L_p(1-n, t; \chi)$. In addition to this result, we have each of the following:

THEOREM 4.11. *Let n, c, k, and k' be positive integers with $k \equiv k'$ $\pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq \left|pq^{-1}F_0\right|_p$. Then*

$$\begin{aligned} &q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \\ &\qquad \equiv q^{-k'}\Delta_c^{k'} \beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'} \beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

These results show that if related congruences hold for

$$\beta_{n,\chi}(0) = -\frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n,\chi_n},$$

then they must also hold for $\beta_{n,\chi}(\tau)$, where τ is any element of $\mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$.

In [9] Granville defined ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$ according to

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n},$$

where the limit is taken in the p-adic sense. In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, $n \in \mathbf{Z}$, $n \geq 1$, and a collection of functions that correspond to generalized Bernoulli polynomials of negative index, $B_{-n,\chi}(t)$, $n \in \mathbf{Z}$, $n \geq 1$. As a result of our definitions, we show that the $B_{-n,\chi}(t)$ are actually power series that can be written in the form

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty}\binom{-n}{m}B_{-n-m,\chi}t^m,$$

converging for $t \in \mathbf{C}_p$, $|t|_p < 1$. We close out by considering some properties of these functions.

2. PRELIMINARIES

The p-adic L-functions, $L_p(s;\chi)$, were first generated by Kubota and Leopoldt for the purpose of finding functions that would serve as analogues of the Dirichlet L-functions in the p-adic number field [14]. They are characterized by the fact that they interpolate a specific expression involving generalized Bernoulli numbers when the variable s is a nonpositive integer. In the following, for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we derive a p-adic function $L_p(s,\tau;\chi)$ that interpolates a specific expression involving generalized