

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS  
**Autor:** BAUM, Paul / CONNES, Alain  
**Kapitel:** 5. The geometric K-theory for  $\pi_0 G$  finite  
**DOI:** <https://doi.org/10.5169/seals-64793>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

5. THE GEOMETRIC  $K$ -THEORY FOR  $\pi_0 G$  FINITE

In this section we shall determine the geometric group  $K^*(X, G)$  whenever  $G$  has only a finite number of connected components. The main point is the existence of a final object (namely  $H \backslash G$ , where  $H$  is the maximal compact subgroup of  $G$ ) in the category of proper  $G$ -manifolds.

Throughout this section  $G$  is a Lie group with a finite number of connected components.  $H$  denotes the maximal compact subgroup of  $G$ . And  $\mathfrak{g}$ ,  $\mathfrak{h}$  are the Lie algebras

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow \mathfrak{h} \backslash \mathfrak{g} \rightarrow 0.$$

Passing to dual spaces (over  $\mathbf{R}$ ):

$$0 \leftarrow \mathfrak{h}^* \leftarrow \mathfrak{g}^* \leftarrow (\mathfrak{h} \backslash \mathfrak{g})^* \leftarrow 0.$$

By the co-adjoint representation  $H$  acts on  $(\mathfrak{h} \backslash \mathfrak{g})^*$

$$(\mathfrak{h} \backslash \mathfrak{g})^* \times H \rightarrow (\mathfrak{h} \backslash \mathfrak{g})^*.$$

Given a  $G$ -manifold  $X$ , let  $H$  act on  $X \times (\mathfrak{h} \backslash \mathfrak{g})^*$  by

$$(x, u)h = (xh, uh)$$

( $x \in X$ ,  $u \in (\mathfrak{h} \backslash \mathfrak{g})^*$ ,  $h \in H$ ).

**PROPOSITION 1.** *For any  $G$ -manifold  $X$  there is a canonical isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h} \backslash \mathfrak{g})^*) \rightarrow K^i(X, G) \quad (i = 0, 1).$$

**REMARK 2.** The isomorphism of the proposition is completely canonical and has no shift of dimension.

**COROLLARY 3.** *Set  $\epsilon = \dim(\mathfrak{h} \backslash \mathfrak{g})$ . If the co-adjoint action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is  $\text{Spin}^c$ , then*

$$K_H^i(X) \cong K^{i+\epsilon}(X, G).$$

*Proof of Corollary 3.* If the action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is  $\text{Spin}^c$ , then the Thom isomorphism [1] applies to give an isomorphism

$$K_H^i(X) \rightarrow K_H^{i+\epsilon}(X \times (\mathfrak{h} \backslash \mathfrak{g})^*).$$

Composing this with the isomorphism of Proposition 1 proves the corollary.  $\square$

REMARK 4. Set  $H \backslash G = \{Hg \mid g \in G\}$ . There is the evident (right) action of  $G$  on  $H \backslash G$

$$(H \backslash G) \times G \rightarrow H \backslash G.$$

The action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is  $\text{Spin}^c$  if and only if  $H \backslash G$  admits a  $G$ -invariant  $\text{Spin}^c$ -structure.

To analyze the case when the action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is not  $\text{Spin}^c$ , fix an  $H$ -invariant Euclidean structure on  $(\mathfrak{h} \backslash \mathfrak{g})^*$ . Proceed as in [15]. Since  $H$  is connected, the co-adjoint representation maps  $H$  into  $\text{SO}(\mathfrak{h} \backslash \mathfrak{g})^*$ . Let  $\text{Spin}(\mathfrak{h} \backslash \mathfrak{g})^*$  be the non-trivial 2-fold covering of  $\text{SO}(\mathfrak{h} \backslash \mathfrak{g})^*$  and form the commutative diagram

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & \text{Spin}(\mathfrak{h} \backslash \mathfrak{g})^* \\ \downarrow & & \downarrow \\ H & \longrightarrow & \text{SO}(\mathfrak{h} \backslash \mathfrak{g})^* \end{array}$$

where  $\tilde{H} = H \times_{\text{SO}(\mathfrak{h} \backslash \mathfrak{g})^*} \text{Spin}(\mathfrak{h} \backslash \mathfrak{g})^*$  is the 2-fold covering of  $H$  obtained by pulling-back the  $\text{Spin}$  covering of  $\text{SO}(\mathfrak{h} \backslash \mathfrak{g})^*$ . There is then ([1]) the Thom isomorphism

$$K_{\tilde{H}}^i(X) \rightarrow K_{\tilde{H}}^{i+\epsilon}(X \times (\mathfrak{h} \backslash \mathfrak{g})^*).$$

Moreover, let  $u \in \tilde{H}$  be the non-identity element of  $\tilde{H}$  which maps to the identity element of  $H$  by the projection  $\tilde{H} \rightarrow H$ . If  $E$  is any  $\tilde{H}$ -vector bundle on  $X$ , there is the direct sum decomposition

$$E = E_+ \oplus E_-$$

where  $E_{\pm} = \{v \in E \mid vu = \pm v\}$ . This leads to a direct sum decomposition of  $K_{\tilde{H}}^*(X)$ :

$$K_{\tilde{H}}^i(X) = \left[ K_{\tilde{H}}^i(X) \right]_+ \oplus \left[ K_{\tilde{H}}^i(X) \right]_- ,$$

where  $\left[ K_{\tilde{H}}^i(X) \right]_{\pm}$  is obtained by only using  $E_{\pm}$ . Note that  $\left[ K_{\tilde{H}}^i(X) \right]_+ \cong K_H^i(X)$ .

COROLLARY 5. For any  $G$ -manifold  $X$ , there is an isomorphism of abelian groups

$$\left[ K_{\tilde{H}}^i(X) \right]_- \rightarrow K^{i+\epsilon}(X, G),$$

$i = 0, 1$ ,  $\epsilon = \dim(\mathfrak{h} \backslash \mathfrak{g})$ .

*Proof.* The Thom isomorphism

$$K_{\tilde{H}}^i(X) \rightarrow K_{\tilde{H}}^{i+\epsilon}(X \times (\mathfrak{h} \backslash \mathfrak{g})^*)$$

gives an isomorphism

$$\left[ K_{\tilde{H}}^i(X) \right]_- \rightarrow K_H^{i+\epsilon}(X \times (\mathfrak{h} \backslash \mathfrak{g})^*).$$

Combining this with the isomorphism of Proposition 1 proves Corollary 5.  $\square$

The essential point in the proof of Proposition 1 is given by

LEMMA 6. *Let  $Z$  be any proper  $G$ -manifold. Then there exists a  $G$ -map from  $Z$  to  $H \backslash G$ .*

*Proof.* Assume for simplicity that  $H \backslash G$  admits a  $G$ -invariant Riemannian metric of non-positive curvature. This is the case if  $G$  is semi-simple [17].

It follows easily from the slice theorem of Palais [23] that  $Z$  can be covered by open sets  $U_0, U_1, U_2, \dots$  such that each  $U_i$  is mapped into itself by  $G$ ,  $\{U_i\}$  is a locally finite cover of  $Z$ , and there exist  $G$ -maps  $f_i: U_i \rightarrow H \backslash G$ . Two points in  $H \backslash G$  are joined by a unique geodesic. Let  $\phi_0: U_0 \cup U_1 \rightarrow \mathbf{R}$ ,  $\phi_1: U_0 \cup U_1 \rightarrow \mathbf{R}$  be a  $C^\infty$  partition of unity on  $U_0 \cup U_1$  subordinate to the covering  $U_0, U_1$  and with each  $\phi_i$  constant on orbits. Then  $\phi_0 f_0 + \phi_1 f_1$  is a  $G$ -map from  $U_0 \cup U_1$  to  $H \backslash G$  where  $(\phi_0 f_0 + \phi_1 f_1)$  means the weighted average (by weights  $\phi_0(x)$ ,  $\phi_1(x)$ ) of  $f_0(x), f_1(x)$  along the unique geodesic joining  $f_0(x)$  and  $f_1(x)$ . Iterating this construction produces the desired  $G$ -map from  $Z$  to  $H \backslash G$ .

The general case has been proved by A. Borel [10].  $\square$

*Proof of Proposition 1.* Let  $(Z, \xi, f)$  be a  $K$ -cocycle for  $(X, G)$ . According to Lemma 6 there is a  $G$ -map  $\theta: Z \rightarrow H \backslash G$ . Let  $h: Z \rightarrow X \times (H \backslash G)$  be

$$h(z) = (fz, \theta z).$$

Form the evident commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \times (H \backslash G) \\ f \searrow & & \swarrow \pi \\ & X & \end{array}$$

where  $\pi: X \times (H \backslash G) \rightarrow X$  is the projection.

Define an isomorphism

$$(1) \quad K^i(X, G) \rightarrow K_G^i(T^*[X \times H \backslash G] \oplus \pi^* T^* X)$$

by

$$(Z, \xi, f) \rightarrow h_!(\xi).$$

Now  $T^*[X \times H \backslash G] \oplus \pi^*T^*X = \pi^*T^*X \oplus \pi^*T^*X \oplus \rho^*T^*(H \backslash G)$ , where  $\rho: X \times H \backslash G \rightarrow H \backslash G$  is the projection.  $\pi^*T^*X \oplus \pi^*T^*X$  has a  $G$ -invariant  $\text{Spin}^c$ -structure. Hence the Thom isomorphism theorem applies to give an isomorphism

$$(2) \quad K_G^i(T^*[X \times H \backslash G] \oplus \pi^*T^*X) \rightarrow K_G^i(\rho^*T^*(H \backslash G)).$$

Next, there is the identification

$$[X \times (\mathfrak{h} \backslash \mathfrak{g})^*] \times_H G = \rho^*T^*(H \backslash G).$$

This identification gives an induction isomorphism

$$(3) \quad K_H^i[X \times (\mathfrak{h} \backslash \mathfrak{g})^*] \rightarrow K_G^i(\rho^*T^*(H \backslash G)).$$

Starting with an  $H$ -vector bundle  $E$  on  $X \times (\mathfrak{h} \backslash \mathfrak{g})^*$  the induction isomorphism takes  $E$  to  $E \times_H G$ . Combining the isomorphisms (1), (2), (3) proves the proposition.  $\square$

REMARK 7. Of special interest is the case when  $X$  is a point. By the above proposition

$$\begin{aligned} K^\epsilon(\cdot, G) &\cong R(\tilde{H})_- \\ K^{1+\epsilon}(\cdot, G) &= 0. \end{aligned}$$

Here  $\epsilon = \dim(\mathfrak{h} \backslash \mathfrak{g})$  and  $R(\tilde{H})_- = K_{\tilde{H}}^0(\cdot)_-$  is the free abelian group with one generator for each irreducible representation of  $\tilde{H}$  which is *not* a representation of  $H$ . If the action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is  $\text{Spin}$ , then there is an identification  $R(\tilde{H})_- = R(H)$ . The second-named author (A. Connes) and independently G. G. Kasparov [20] have conjectured that Dirac induction gives an isomorphism

$$\begin{aligned} K_\epsilon[C^*G] &\cong R(\tilde{H})_- \\ K_{1+\epsilon}[C^*G] &= 0. \end{aligned}$$

For connected complex semi-simple groups M. Pennington and R. Plymen [25], [28], have verified this conjecture. These results of M. Pennington and R. Plymen combined with the proposition of this section verify the isomorphism conjecture stated in §2 above in a number of interesting cases. Note that (due to the proposition of this section) the Connes-Kasparov conjecture on  $K_*C^*G$  is a special case of the isomorphism conjecture of §2.

Let  $G$  be a connected semi-simple Lie group with finite center. The lemma of this section elucidates the role of  $H \backslash G$  in the Atiyah-Schmid geometric construction of the discrete series [4]. Atiyah and Schmid obtain the discrete series representations by using the Dirac operator on  $H \backslash G$ . As noted in the introduction  $K_0[C^*G]$  contains a free abelian group with one generator for each (irreducible) discrete series representation. By the lemma, however, all of  $K^*(\cdot, G)$  is obtained from  $H \backslash G$ . If (as conjectured in §2 above)  $K^*(\cdot, G) \cong K_*(C^*G)$ , then not only the discrete series, but *all* of  $K_*(C^*G)$  can be obtained from  $H \backslash G$ .

At this juncture one might ask, “*Why not simply define  $K^i(X, G) = K_H^i(X)$  ?*” We believe that there are compelling reasons for not doing this. First, this misses the dimension-shift by  $\epsilon = \dim(H \backslash G)$ . Second, this overlooks the issue of whether or not the action of  $H$  on  $(\mathfrak{h} \backslash \mathfrak{g})^*$  is  $\text{Spin}^c$ . Third, this greatly obscures the relation of  $K$ -theory to index theory. Finally, in the case of discrete groups and foliations there is no maximal compact subgroup so that if this were done there would be no unified theory for Lie groups, discrete groups, and foliations.

## 6. DISCRETE GROUPS: CHERN CHARACTER

In this section  $G$  is a discrete group which is either finite or countable infinite. For a  $G$ -manifold  $X$ ,  $K^*(X, G)$  was defined in §2 above. As in §3 there is the natural map

$$K_*^\tau([EG \times X]/G) \rightarrow K^*(X, G),$$

where  $\tau = [EG \times T^*X]/G$ .

PROPOSITION 1. *Let  $G$  be a discrete group and  $X$  a  $G$ -manifold. Then*

$$K_*^\tau([EG \times X]/G) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K^*(X, G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*is injective.*

REMARK 2. When  $X$  is a point, Proposition 1 asserts that

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K^*(\cdot, G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*is injective.*