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**Autor:** BAUM, Paul / CONNES, Alain  
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Addition in  $K^*(X, G)$  is given by disjoint union of  $K$ -cocycles. Further,

$$K^*(X, G) = K^0(X, G) \oplus K^1(X, G),$$

where  $K^i(X, G)$  is the subgroup of  $K^*(X, G)$  determined by all  $K$ -cocycles  $(Z, \xi, f)$  with  $\xi \in V_G^i(T^*Z \oplus f^*T^*X)$ . The natural homomorphism of abelian groups

$$K^i(X, G) \rightarrow K_i[C_0(X) \rtimes G]$$

is defined by

$$(Z, \xi, f) \mapsto \mu(Z, \xi, f).$$

CONJECTURE. *For any  $G$ -manifold  $X$ ,  $\mu: K^i(X, G) \rightarrow K_i[C_0(X) \rtimes G]$  is an isomorphism.*

This conjecture is known to be true if  $X$  is a proper  $G$ -manifold. If  $X$  is proper there is a commutative diagram

$$\begin{array}{ccc} K^*(X, G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ & i_t \searrow & \swarrow \alpha \\ & K_G^*(X) & \end{array}$$

in which each arrow is an isomorphism.  $i_t: K^*(X, G) \rightarrow K_G^*(X)$  maps a  $K$ -cocycle  $(Z, \xi, f)$  to its topological index, and  $\alpha \circ \mu: K^*(X, G) \rightarrow K_G^*(X)$  maps a  $K$ -cocycle  $(Z, \xi, f)$  to its analytic index. If  $G$  is compact then any  $G$ -manifold is proper and commutativity of the diagram is equivalent to the Atiyah-Singer index theorems of [6], [7], [8].

### 3. HOMOTOPY QUOTIENT

Let  $W$  be a topological space.  $V^0(W)$  denotes the collection of all complex vector bundles  $(E_0, E_1, \sigma)$  on  $W$  with compact support. Thus  $E_0, E_1$  are complex vector bundles on  $W$  and  $\sigma: E_0 \rightarrow E_1$  is a morphism of complex vector bundles with  $\text{Support}(\sigma)$  compact, where

$$\text{Support}(\sigma) = \{p \in W \mid \sigma: E_{0p} \rightarrow E_{1p} \text{ is not an isomorphism}\}.$$

Also  $V^1(W) = V^0(W \times \mathbf{R})$ .

Suppose given an  $\mathbf{R}$ -vector bundle  $F$  on  $W$ . Following [9], a *twisted* by  $F$   $K$ -cycle on  $W$  is a triple  $(M, \xi, \phi)$  such that:

- (1)  $M$  is a  $C^\infty$ -manifold without boundary;
- (2)  $\phi: M \rightarrow W$  is a continuous map from  $M$  to  $W$ ;
- (3)  $\xi \in V^*(T^*M \oplus \phi^*F)$ .

As in [9] an equivalence relation is imposed on these twisted by  $F$   $K$ -cycles to obtain the twisted by  $F$   $K$ -homology of  $W$ :

$$K_*^F(W) = K_0^F(W) \oplus K_1^F(W).$$

$K_1^F(W)$  is the subgroup determined by all  $(M, \xi, \phi)$  with  $\xi \in V^i(T^*M \oplus \phi^*F)$ . If  $F$  has a  $\text{Spin}^c$ -structure then  $K_*^F(W)$  is isomorphic to  $K_*(W)$ , the  $K$ -homology of  $W$ .

With  $G$  as in §2 above, let  $EG$  be a contractible space on which  $G$  acts freely

$$EG \times G \rightarrow EG.$$

Given a  $G$ -manifold  $X$ , let  $G$  act on  $EG \times X$  by

$$(p, x)g = (pg, xg)$$

( $p \in EG, x \in X, g \in G$ ). The quotient space  $[EG \times X]/G$  will be referred to as the homotopy quotient. Since  $T^*X$  is a  $G$ -vector bundle on  $X$ , the quotient  $[EG \times T^*X]/G$  is a vector bundle on  $[EG \times X]/G$ . Denote this vector bundle by  $\tau$  and consider the twisted by  $\tau$   $K$ -homology  $K_*^\tau([EG \times X]/G)$ . There is a map

$$K_*^\tau([EG \times X]/G) \rightarrow K^*(X, G).$$

This map is not quite canonical. First an orientation must be chosen for the Lie algebra of  $G$ , so assume that such an orientation has been chosen.

Let  $(M, \xi, \phi)$  be a twisted by  $\tau$   $K$ -cycle on  $[EG \times X]/G$ . Now  $EG \times X$  is the total space of a principal  $G$ -bundle over  $[EG \times X]/G$  and this principal bundle can be pulled back via  $\phi$  to yield a principal bundle  $Z$  over  $M$

$$\begin{array}{ccc} EG \times X & \xleftarrow{\tilde{\phi}} & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \xleftarrow{\phi} & M. \end{array}$$

Let  $\pi: EG \times X \rightarrow X$  be the projection and set  $f = \pi \circ \tilde{\phi}$ ,

$$f: Z \rightarrow X.$$

$\xi \in V^*(T^*M \oplus \phi^*\tau)$  lifts to give  $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$ . Denote the bundle along the fibres of  $\rho: Z \rightarrow M$  by  $F$ . This is a trivial vector bundle since,

for each  $z \in Z$ ,  $F_z$  is canonically isomorphic to the Lie algebra of  $G$ . Using the orientation of this Lie algebra,  $F$  has a  $G$ -invariant  $\text{Spin}^c$ -structure so that  $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$  determines  $\eta \in V_G^*(F \oplus \rho^*T^*M \oplus f^*T^*X)$ . Now  $F \oplus \rho^*T^*M = T^*Z$ , so  $(Z, \eta, f)$  is a  $K$ -cocycle for  $(X, G)$ . The map

$$K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$$

is:

$$(M, \xi, \phi) \mapsto (Z, \eta, f).$$

This map has a dimension-shift in it. Set  $\epsilon = \dim(G)$ . Then with addition of indices mod 2 this map takes  $K_i^T([EG \times X]/G)$  to  $K^{i+\epsilon}(X, G)$ .

LEMMA 1. *If  $G$  is torsion free then  $K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$  is an isomorphism.*

*Proof.* Let  $(Z, \xi, f)$  be a  $K$ -cocycle for  $(X, G)$ . The action of  $G$  on  $Z$  is proper, so each isotropy group is compact. Since  $G$  is assumed to be torsion free this implies that the action of  $G$  on  $Z$  is free. Hence  $Z$  is a  $G$ -principal bundle over  $G/Z$ , and thus  $Z$  maps equivariantly to  $EG$ . Combining this with  $f: Z \rightarrow X$  we obtain a commutative diagram

$$\begin{array}{ccc} EG \times X & \longleftarrow & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \longleftarrow & Z/G. \end{array}$$

Denote the map of  $Z/G$  to  $[EG \times X]/G$  by  $\phi$ . Then  $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$  determines  $\xi' \in V_G^*(\rho^*T^*(Z/G) \oplus f^*T^*X)$ . Since the action of  $G$  on  $Z$  is free  $\xi'$  descends to give  $\theta \in V^*(T^*(Z/G) \oplus \tau)$ . Then

$$(Z, \xi, f) \rightarrow (Z/G, \theta, \phi)$$

maps  $K^*(X, G)$  to  $K_*^T([EG \times X]/G)$  and provides an inverse to the map  $K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$ .  $\square$

REMARK 2. If  $G$  is the trivial one-element group then the isomorphism of the lemma becomes

$$K_*^{T^*X}(X) \cong K^*(X).$$

If  $X$  is a  $\text{Spin}^c$ -manifold then  $K_*^{T^*X}(X) \cong K_*(X)$ , so that in this case the isomorphism of the lemma becomes the Poincaré duality isomorphism  $K_*(X) \cong K^*(X)$ .

When  $G$  has torsion, the map  $K_*^{\tau}([EG \times X]/G) \rightarrow K^*(X, G)$  can fail to be an isomorphism. The simplest example of this is obtained by taking  $X$  to be a point and  $G = \mathbf{Z}/2\mathbf{Z}$ .

When  $G$  has torsion,  $K_*^{\tau}([EG \times X]/G)$  appears to be only a first approximation to  $K^*(X, G)$  and  $K_*[C_0(X) \rtimes G]$ . The key point is that when  $G$  has torsion, there will be proper  $G$ -manifolds on which the  $G$ -action is not free.

#### 4. SOLVABLE SIMPLY CONNECTED LIE GROUPS

The conjecture stated in §2 above is verified for (connected) solvable simply connected Lie groups by

**PROPOSITION 1.** *Let  $G$  be a (connected) solvable simply connected Lie group, and let  $X$  be a  $G$ -manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*(X, G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ \downarrow & & \downarrow \\ K^*(X) & \longrightarrow & K_*[C_0(X)] \end{array}$$

*in which each arrow is an isomorphism.*

The proof depends on

**LEMMA 2.** *Let  $G$  be a (connected) solvable simply connected Lie group, and let  $Z$  be a proper  $G$ -manifold. Then there exists a  $G$ -map from  $Z$  to  $G$ .*

*Proof of Lemma 2.* Since the action of  $G$  on  $Z$  is proper all isotropy groups are compact.  $G$  has no non-trivial compact subgroups, so the action of  $G$  on  $Z$  is free. Therefore  $Z$  is a principal  $G$ -bundle with base  $Z/G$ . As  $G$  is itself a contractible space on which  $G$  acts freely, there is a  $G$ -map from  $Z$  to  $G$ .  $\square$

*Proof of Proposition 1.* In the diagram of the proposition the right vertical arrow is the Thom isomorphism of [13]. The lower horizontal arrow is the standard isomorphism which is valid for any locally compact Hausdorff topological space.