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## A FREE GROUP ACTING ON $\mathbf{Z}^2$ WITHOUT FIXED POINTS

by SATÔ Kenzi

ABSTRACT. The group of all orientation-preserving affine transformations of the plane has a non-abelian free subgroup which stabilizes  $\mathbf{Z}^2$  and which acts on  $\mathbf{Z}^2$  without non-trivial fixed points.

### INTRODUCTION

Let  $G$  be a group acting on a non-empty set  $X$ . The following two conditions are known to be equivalent (see [D], and Theorems 4.5 and 4.8 in [W]):

- (a) *there exists a non-abelian free subgroup of  $G$  whose action on  $X$  is locally commutative;*
- (b) *there exists a  $G$ -paradoxical decomposition of  $X$  using 4 pieces, namely a partition of  $X$  in parts  $P_0, P_1, P_2, P_3$  and elements  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  in  $G$  such that*

$$X = P_0 \sqcup P_1 \sqcup P_2 \sqcup P_3 = \alpha_0(P_0) \sqcup \alpha_1(P_1) = \alpha_2(P_2) \sqcup \alpha_3(P_3).$$

Moreover, in the situation of (b), it can be shown that the subgroup of  $G$  generated by  $\alpha_0^{-1}\alpha_1$  and  $\alpha_2^{-1}\alpha_3$  is free of rank 2. (The symbol  $\sqcup$  denotes disjoint union. Recall that an action of a group  $H$  on  $X$  is *locally commutative* if the stabilizer  $\{h \in H \mid h(x) = x\}$  is commutative for all  $x \in X$ , i.e. if two elements of  $H$  which have a common fixed point commute; trivial examples of locally commutative actions are actions *without non-trivial fixed points*, for which  $\{h \in H \mid h(x) = x\}$  is reduced to  $\{1\}$  for all  $x \in X$ .)

For example, the group  $SO_3(\mathbf{R})$  of rotations of the unit sphere  $S^2$  has such a free subgroup: this was discovered by F. Hausdorff (see, e.g., [Š], or Theorem 2.1 in [W]). It implies the following result, for which we refer to [BT] and Theorem 3.11 in [W]; we denote by  $SG_3(\mathbf{R})$  the group of all orientation-preserving isometries of  $\mathbf{R}^3$ .

**THE BANACH-TARSKI PARADOX.** *Any two bounded subsets  $U$  and  $V$  of the 3-dimensional Euclidean space  $\mathbf{R}^3$  with non-empty interiors are  $SG_3(\mathbf{R})$ -equidecomposable. In other words, one can partition  $U$  into a finite number of pieces and reconstruct  $V$  from the same number of respectively  $SG_3(\mathbf{R})$ -congruent pieces.*

The Banach-Tarski paradox holds similarly for higher dimensional Euclidean spaces, but not for  $\mathbf{R}$  and  $\mathbf{R}^2$ ; the reason is that neither  $SG_1(\mathbf{R})$  nor  $SG_2(\mathbf{R})$ , which are soluble groups, contain free subgroups of rank 2. (There are other known examples of free groups acting without non-trivial fixed points on familiar spaces. See e.g., [B], [DS], and [S2]. The proof of the Banach-Tarski paradox requires the axiom of choice, because the proof of the equivalence of conditions (a) and (b) requires it. But similar paradoxes hold for rational spheres of the form  $(\sqrt{q}S^2) \cap \mathbf{Q}^3$ , as can be shown *without* the axiom of choice from the countability of rational spheres. See [S1], and [S3].) In dimension 2, von Neumann has exhibited a Banach-Tarski paradox with respect to the group  $SA_2(\mathbf{R})$  of affine transformations of  $\mathbf{R}^2$  that preserve area and orientation ([V], and Theorem 7.3 of [W]). The following problem was raised in [MW]; see also the discussion which follows Proposition 7.1 in [W].

**PROBLEM ([MW], [W]).** *Does  $SA_2(\mathbf{R})$  contain a free subgroup of rank 2 whose action on  $\mathbf{R}^2$  is locally commutative?*

Indeed, these authors asked more specifically if the group generated by

$$\alpha: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\beta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies the requirements of the problem. We observe here that the answer is “no”, because both  $\alpha^{-2}\beta^2$  and  $\alpha^{-1}\beta^{-1}\alpha\beta$  fix the origin.

Though we cannot solve the above problem, the purpose of this note is to show that, if one replaces  $\mathbf{R}^2$  by  $\mathbf{Z}^2$ , the new problem has a positive solution. In fact, we will prove the following result, which shows somewhat more, namely that the action on  $\mathbf{Z}^2$  may be an action without non-trivial fixed points, rather than only locally commutative. We denote by  $\text{SA}_2(\mathbf{Z})$  the group of all transformations  $\vec{x} \mapsto A\vec{x} + \vec{a}$  of  $\mathbf{Z}^2$ , with  $A \in \text{SL}_2(\mathbf{Z})$  and  $\vec{a} \in \mathbf{Z}^2$ .

**THEOREM.** *The group  $\text{SA}_2(\mathbf{Z})$  has a free subgroup  $F_2$  of rank 2 which acts on  $\mathbf{Z}^2$  without non-trivial fixed points, namely the subgroup generated by*

$$\zeta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\eta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The theorem implies the existence of a partition of  $\mathbf{Z}^2$  into three pieces  $P$ ,  $Q$  and  $R$  such that the six pieces  $P$ ,  $Q$ ,  $R$ ,  $P \sqcup Q$ ,  $Q \sqcup R$ ,  $R \sqcup P$  are pairwise  $F_2$ -congruent, without the axiom of choice ([S0], and Corollary 4.12 in [W]).

As observed in the discussion which follows Proposition 7.1 in [W], it is known that the above theorem does not carry over to  $\mathbf{R}^2$ ; more precisely, it is known that a subgroup of  $\text{SA}_2(\mathbf{R})$  which acts on  $\mathbf{R}^2$  without non-trivial fixed points is soluble, and consequently does not contain non-commutative free subgroups.

## PROOF OF THE MAIN RESULT

Recall that a matrix in  $\text{SL}_2(\mathbf{Z})$  is *hyperbolic* if the absolute value of its trace is strictly larger than 2, or equivalently if it has an eigenvalue of absolute value strictly larger than 1.

**LEMMA 0.** *The subgroup of  $\text{SL}_2(\mathbf{Z})$  generated by*

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix}$$

*is free of rank 2 and all its elements distinct from the identity are hyperbolic.*

*Proof.* It is well-known that the subgroup of  $SL_2(\mathbf{Z})$  generated by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

is free of rank 2, and that all its elements distinct from the identity are hyperbolic. (See Appendix B in [K], [Ma], [MW], [N], or the proof of Theorem 6.8 in [W].) The lemma follows, because

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^2$$

(see for example Exercice 12 of Section 1.4 in [MKS]).  $\square$

The following is elementary linear algebra.

LEMMA 1. For  $A \in SL_2(\mathbf{R})$  with  $\text{tr} A \neq 2$  and for  $\vec{a} \in \mathbf{R}^2$ , the affine transformation

$$\begin{cases} \mathbf{R}^2 & \rightarrow \mathbf{R}^2 \\ \vec{x} & \mapsto A\vec{x} + \vec{a} \end{cases}$$

has a unique fixed point.

Our preparations are complete.

*Proof of the main theorem.* The two transformations  $\zeta$  and  $\eta$  of our main result generate a group which is free of rank 2, by Lemma 0. As both these transformations fix the point

$$\begin{pmatrix} 2/3 \\ -5/3 \end{pmatrix} \in \mathbf{R}^2,$$

each element of the group they generate fix the same point. As this point is not in  $\mathbf{Z}^2$ , the theorem follows by Lemma 1.  $\square$

REMARK. Let  $\alpha, \beta \in SA_2(\mathbf{Z})$  be as in the introduction. Then we can prove the main theorem by using the group generated by  $\alpha\beta^{-1}\alpha\beta^{-2}\alpha$  and  $\beta\alpha^{-1}\beta\alpha^{-2}\beta$ , because the transformations

$$\alpha\beta^{-1}\alpha\beta^{-2}\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -17 \\ -13 \end{pmatrix}$$

and

$$\beta\alpha^{-1}\beta\alpha^{-2}\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -13 \\ -17 \end{pmatrix}$$

have a common fixed point  $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$  in  $\mathbf{R}^2$  and the subgroup of  $\mathrm{SL}_2(\mathbf{Z})$  generated by

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$$

is free of rank 2 and all its elements distinct from the identity are hyperbolic. See [K] and the following calculations:

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} = tu(tu^{-1})^3(tu)^2tu^{-1}tu, \\ \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} = tu^{-1}(tu)^3(tu^{-1})^2tutu^{-1},$$

where

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

with  $\langle t, u \rangle / \{\pm 1\} \cong \mathbf{Z}_2 * \mathbf{Z}_3$ .

The referee suggested to the author that the following could be shown (without the axiom of choice).

**COROLLARY.** *There exists a subset  $E_1$  of  $\mathbf{Z}^2$  such that, for every finite subset  $F$  of  $\mathbf{Z}^2$ , the symmetric difference of  $E_1$  and  $F$  is congruent to  $E_1$  relative to the group  $\mathrm{SA}_2(\mathbf{Z})$ .*

*Proof.* This is a consequence of our main result and of Theorem 2 in [My] ( $S = \mathbf{Z}^2$ ,  $G = \langle \zeta, \eta \rangle$ ,  $M = \{\zeta\eta, \zeta^2\eta^2, \zeta^3\eta^3, \dots\}$ ,  $\mathbf{F} = \{F \subseteq \mathbf{Z}^2 \mid F \text{ is finite}\}$ ).

□

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