

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FREE GROUP ACTING ON Z^2 WITHOUT FIXED POINTS
Autor: Kenzi, Satô
DOI: <https://doi.org/10.5169/seals-64445>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A FREE GROUP ACTING ON \mathbf{Z}^2 WITHOUT FIXED POINTS

by SATÔ Kenzi

ABSTRACT. The group of all orientation-preserving affine transformations of the plane has a non-abelian free subgroup which stabilizes \mathbf{Z}^2 and which acts on \mathbf{Z}^2 without non-trivial fixed points.

INTRODUCTION

Let G be a group acting on a non-empty set X . The following two conditions are known to be equivalent (see [D], and Theorems 4.5 and 4.8 in [W]):

- (a) *there exists a non-abelian free subgroup of G whose action on X is locally commutative;*
- (b) *there exists a G -paradoxical decomposition of X using 4 pieces, namely a partition of X in parts P_0, P_1, P_2, P_3 and elements $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ in G such that*

$$X = P_0 \sqcup P_1 \sqcup P_2 \sqcup P_3 = \alpha_0(P_0) \sqcup \alpha_1(P_1) = \alpha_2(P_2) \sqcup \alpha_3(P_3).$$

Moreover, in the situation of (b), it can be shown that the subgroup of G generated by $\alpha_0^{-1}\alpha_1$ and $\alpha_2^{-1}\alpha_3$ is free of rank 2. (The symbol \sqcup denotes disjoint union. Recall that an action of a group H on X is *locally commutative* if the stabilizer $\{h \in H \mid h(x) = x\}$ is commutative for all $x \in X$, i.e. if two elements of H which have a common fixed point commute; trivial examples of locally commutative actions are actions *without non-trivial fixed points*, for which $\{h \in H \mid h(x) = x\}$ is reduced to $\{1\}$ for all $x \in X$.)

For example, the group $SO_3(\mathbf{R})$ of rotations of the unit sphere S^2 has such a free subgroup: this was discovered by F. Hausdorff (see, e.g., [S], or Theorem 2.1 in [W]). It implies the following result, for which we refer to [BT] and Theorem 3.11 in [W]; we denote by $SG_3(\mathbf{R})$ the group of all orientation-preserving isometries of \mathbf{R}^3 .

THE BANACH-TARSKI PARADOX. *Any two bounded subsets U and V of the 3-dimensional Euclidean space \mathbf{R}^3 with non-empty interiors are $SG_3(\mathbf{R})$ -equidecomposable. In other words, one can partition U into a finite number of pieces and reconstruct V from the same number of respectively $SG_3(\mathbf{R})$ -congruent pieces.*

The Banach-Tarski paradox holds similarly for higher dimensional Euclidean spaces, but not for \mathbf{R} and \mathbf{R}^2 ; the reason is that neither $SG_1(\mathbf{R})$ nor $SG_2(\mathbf{R})$, which are soluble groups, contain free subgroups of rank 2. (There are other known examples of free groups acting without non-trivial fixed points on familiar spaces. See e.g., [B], [DS], and [S2]. The proof of the Banach-Tarski paradox requires the axiom of choice, because the proof of the equivalence of conditions (a) and (b) requires it. But similar paradoxes hold for rational spheres of the form $(\sqrt{q} S^2) \cap \mathbf{Q}^3$, as can be shown without the axiom of choice from the countability of rational spheres. See [S1], and [S3].) In dimension 2, von Neumann has exhibited a Banach-Tarski paradox with respect to the group $SA_2(\mathbf{R})$ of affine transformations of \mathbf{R}^2 that preserve area and orientation ([V], and Theorem 7.3 of [W]). The following problem was raised in [MW]; see also the discussion which follows Proposition 7.1 in [W].

PROBLEM ([MW], [W]). *Does $SA_2(\mathbf{R})$ contain a free subgroup of rank 2 whose action on \mathbf{R}^2 is locally commutative?*

Indeed, these authors asked more specifically if the group generated by

$$\alpha: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\beta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies the requirements of the problem. We observe here that the answer is “no”, because both $\alpha^{-2}\beta^2$ and $\alpha^{-1}\beta^{-1}\alpha\beta$ fix the origin.

Though we cannot solve the above problem, the purpose of this note is to show that, if one replaces \mathbf{R}^2 by \mathbf{Z}^2 , the new problem has a positive solution. In fact, we will prove the following result, which shows somewhat more, namely that the action on \mathbf{Z}^2 may be an action without non-trivial fixed points, rather than only locally commutative. We denote by $\text{SA}_2(\mathbf{Z})$ the group of all transformations $\vec{x} \mapsto A\vec{x} + \vec{a}$ of \mathbf{Z}^2 , with $A \in \text{SL}_2(\mathbf{Z})$ and $\vec{a} \in \mathbf{Z}^2$.

THEOREM. *The group $\text{SA}_2(\mathbf{Z})$ has a free subgroup F_2 of rank 2 which acts on \mathbf{Z}^2 without non-trivial fixed points, namely the subgroup generated by*

$$\zeta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\eta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The theorem implies the existence of a partition of \mathbf{Z}^2 into three pieces P , Q and R such that the six pieces P , Q , R , $P \sqcup Q$, $Q \sqcup R$, $R \sqcup P$ are pairwise F_2 -congruent, without the axiom of choice ([S0], and Corollary 4.12 in [W]).

As observed in the discussion which follows Proposition 7.1 in [W], it is known that the above theorem does not carry over to \mathbf{R}^2 ; more precisely, it is known that a subgroup of $\text{SA}_2(\mathbf{R})$ which acts on \mathbf{R}^2 without non-trivial fixed points is soluble, and consequently does not contain non-commutative free subgroups.

PROOF OF THE MAIN RESULT

Recall that a matrix in $\text{SL}_2(\mathbf{Z})$ is *hyperbolic* if the absolute value of its trace is strictly larger than 2, or equivalently if it has an eigenvalue of absolute value strictly larger than 1.

LEMMA 0. *The subgroup of $\text{SL}_2(\mathbf{Z})$ generated by*

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix}$$

is free of rank 2 and all its elements distinct from the identity are hyperbolic.

Proof. It is well-known that the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ generated by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

is free of rank 2, and that all its elements distinct from the identity are hyperbolic. (See Appendix B in [K], [Ma], [MW], [N], or the proof of Theorem 6.8 in [W].) The lemma follows, because

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^2$$

(see for example Exercice 12 of Section 1.4 in [MKS]). \square

The following is elementary linear algebra.

LEMMA 1. *For $A \in \mathrm{SL}_2(\mathbf{R})$ with $\mathrm{tr} A \neq 2$ and for $\vec{a} \in \mathbf{R}^2$, the affine transformation*

$$\begin{cases} \mathbf{R}^2 & \rightarrow \mathbf{R}^2 \\ \vec{x} & \mapsto A\vec{x} + \vec{a} \end{cases}$$

has a unique fixed point.

Our preparations are complete.

Proof of the main theorem. The two transformations ζ and η of our main result generate a group which is free of rank 2, by Lemma 0. As both these transformations fix the point

$$\begin{pmatrix} 2/3 \\ -5/3 \end{pmatrix} \in \mathbf{R}^2,$$

each element of the group they generate fix the same point. As this point is not in \mathbf{Z}^2 , the theorem follows by Lemma 1. \square

REMARK. Let $\alpha, \beta \in \mathrm{SA}_2(\mathbf{Z})$ be as in the introduction. Then we can prove the main theorem by using the group generated by $\alpha\beta^{-1}\alpha\beta^{-2}\alpha$ and $\beta\alpha^{-1}\beta\alpha^{-2}\beta$, because the transformations

$$\alpha\beta^{-1}\alpha\beta^{-2}\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -17 \\ -13 \end{pmatrix}$$

and

$$\beta\alpha^{-1}\beta\alpha^{-2}\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -13 \\ -17 \end{pmatrix}$$

have a common fixed point $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ in \mathbf{R}^2 and the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ generated by

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$$

is free of rank 2 and all its elements distinct from the identity are hyperbolic. See [K] and the following calculations:

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} = tu(tu^{-1})^3(tu)^2tu^{-1}tu,$$

$$\begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} = tu^{-1}(tu)^3(tu^{-1})^2tutu^{-1},$$

where

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

with $\langle t, u \rangle / \{\pm 1\} \cong \mathbf{Z}_2 * \mathbf{Z}_3$.

The referee suggested to the author that the following could be shown (without the axiom of choice).

COROLLARY. *There exists a subset E_1 of \mathbf{Z}^2 such that, for every finite subset F of \mathbf{Z}^2 , the symmetric difference of E_1 and F is congruent to E_1 relative to the group $\mathrm{SA}_2(\mathbf{Z})$.*

Proof. This is a consequence of our main result and of Theorem 2 in [My] ($S = \mathbf{Z}^2$, $G = \langle \zeta, \eta \rangle$, $M = \{\zeta\eta, \zeta^2\eta^2, \zeta^3\eta^3, \dots\}$, $\mathbf{F} = \{F \subseteq \mathbf{Z}^2 \mid F \text{ is finite}\}$). \square

REFERENCES

- [BT] BANACH, S. and A. TARSKI. Sur la décomposition des ensembles de points en parties respectivement congruentes. *Fund. Math.* 6 (1924), 244–277.
- [B] BOREL, A. On free subgroups of semi-simple groups. *L'Enseignement Math.* 29 (1983), 151–164.
- [D] DEKKER, Th. J. Decompositions of sets and spaces I. *Indag. Math.* 18 (1956), 581–589.
- [DS] DELIGNE, P. and D. SULLIVAN. Division algebras and the Hausdorff-Banach-Tarski paradox. *L'Enseignement Math.* 29 (1983), 145–150.
- [K] KUROSH, A. G. *The Theory of Groups*, vol. 2. Chelsea Publishing Company, New York, 1956.

- [Ma] MAGNUS, W. Rational representations of Fuchsian groups and non-parabolic subgroups of the modular group. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II* (1973), 179–189.
- [MKS] MAGNUS, W., A. KARASS and D. SOLITAR. *Combinatorial Group Theory*. Interscience, New York, 1966.
- [My] MYCIELSKI, J. About sets invariant with respect to denumerable changes. *Fund. Math.* 45 (1958), 296–305.
- [MW] MYCIELSKI, J. and S. WAGON. Large free groups of isometries and their geometrical uses. *L'Enseignement Math.* 30 (1984), 247–267.
- [N] NEUMANN, B. Über ein gruppentheoretisch-arithmetisches Problem. *Sonderausgabe aus den Sitzungsberichten der Preußischen Akademie der Wissenschaften. Phys.-Math. Klasse X* (1933), 427–444.
- [S0] SATÔ, K. A Hausdorff decomposition on a countable subset of S^2 without the axiom of choice. *Math. Japon.* 44 (1996), 307–312.
- [S1] —— A free group acting without fixed points on the rational unit sphere. *Fund. Math.* 148 (1995), 63–69.
- [S2] —— A free group of rotations with rational entries on the 3-dimensional unit sphere. *Nihonkai Math. J.* 8 (1997), 91–94.
- [S3] —— Free groups acting without fixed points on rational spheres. *Acta Arith.* 85 (1998), 135–140.
- [Ś] ŚWIERCZKOWSKI, S. On a free group of rotations of the Euclidean space. *Indag. Math.* 20 (1958), 376–378.
- [V] VON NEUMANN, J. Zur allgemeinen Theorie des Maßes. *Fund. Math.* 13 (1929), 73–116.
- [W] WAGON, S. *The Banach-Tarski Paradox*. Cambridge Univ. Press, Cambridge-New York, 1985.

(Reçu le 27 octobre 1998)

Satô Kenzi

Department of Mathematics
 Faculty of Engineering
 Tamagawa University
 6-1-1, Tamagawa-Gakuen, Machida
 Tokyo 194-8610
 Japan
 e-mail : kenzi@eng.tamagawa.ac.jp