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Artikel: TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

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6. Applications

LEMMA 13. *Let M be a closed hyperbolic surface of genus g which has $2g-2$ simple closed geodesics $u_1, \ldots, u_{2g-2}$ which all intersect in the same point Q and intersect in no other point. Then M has simple closed curves u_{2g-1} and u_{2g}, passing through Q, such that the curves u_i intersect in no other point than Q, $i = 1, \ldots, 2g$. Moreover, u_{2g-1} and u_g can be chosen such that*

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon $P(g)$.

Proof. Cut M along u_1, the result is a hyperbolic surface M_1 with boundary and genus $g-1$, the boundary consists of two simple closed geodesics v_1 and w_1. Cut M_1 along u_2, the result is a hyperbolic surface M_2 with one boundary component v_2 and genus $g-1$. Now cut M along all $2g-2$ simple closed geodesics $u_1, \ldots, u_{2g-2}$. By induction, the result is a hyperbolic surface M_{2g-2} with one boundary component v and genus 1. More precisely, the boundary v is piecewise geodesic with $4g-4$ pieces and we may assume that the notation is chosen such that these pieces appear on v in the order (the pieces are called like the corresponding closed curves) $u_1, u_2, \ldots, u_{2g-2}, u_1, u_2, \ldots, u_{2g-2}$ (note that closed geodesics intersect transversally). Denote by S and S' the two copies of Q on v between u_1 and u_{2g-2}. Let u_{2g-1} be a simple geodesic in M_{2g-2} which joins S and S' such that u_{2g-1} is not homotopic to a part of v. Cut M_{2g-2} along u_{2g-1}. The result is a hyperbolic surface M_{2g-1} of genus zero with two boundary components w and w' which both consist of $2g-1$ geodesic pieces in the order $u_1, u_2, \ldots, u_{2g-2}, u_{2g-1}$. Denote by R and R' the copies of Q between u_1 and u_{2g-1} on w and w', respectively. Let u_{2g} be a simple geodesic in M_{2g-1} which joins R and R', u_{2g} can be chosen such that when we cut M_{2g-1} along u_{2g}, then we obtain the interior of a canonical polygon as desired. □

DEFINITION. A *hyperelliptic surface* is a closed hyperbolic surface of genus g which has an isometry ϕ with $\phi^2 = id$ and with exactly $2g+2$ fixed points.

In [14], the equivalence of (i) and (ii) of the following theorem was first proved. With the approach chosen here, we can give a third equivalence and a different proof.

THEOREM 14. *Let M be a closed hyperbolic surface M of genus g. Then the following conditions are equivalent.*

(i) *M is hyperelliptic.*

(ii) *M has a set of at least $2g-2$ simple closed geodesics which all intersect in the same point and intersect in no other point.*

(iii) *M has a corresponding canonical polygon with equal opposite angles ($\alpha_i = \alpha_{2g+i}$, $i = 1, \dots, 2g$).*

Proof. I shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Let M be hyperelliptic. Let R_i, $i = 1, \dots, 2g+2$, be the fixed points of a hyperelliptic involution ϕ. Let c_1 be a simple geodesic segment from R_1 to R_2. Then $c_1 \cup \phi(c_1)$ is a simple closed geodesic u_1 since $\phi^2 = id$. It also follows that on u_1, there are only two fixed points of ϕ and that $M_1 = M \setminus u_1$ is connected. Therefore, we can choose a simple geodesic segment c_2 from R_1 to R_3 which intersects u_1 only in R_1. By the same argument as above, $c_2 \cup \phi(c_2)$ is a simple closed geodesic, $M_2 = M \setminus (u_1 \cup u_2)$ is connected and on $u_1 \cup u_2$, there are only three fixed points of ϕ. Continuing this construction we can find simple closed geodesics $u_1, \dots, u_{2g-2}$ which all intersect in R_1 and in no other point. This proves (i) $\Rightarrow$ (ii).

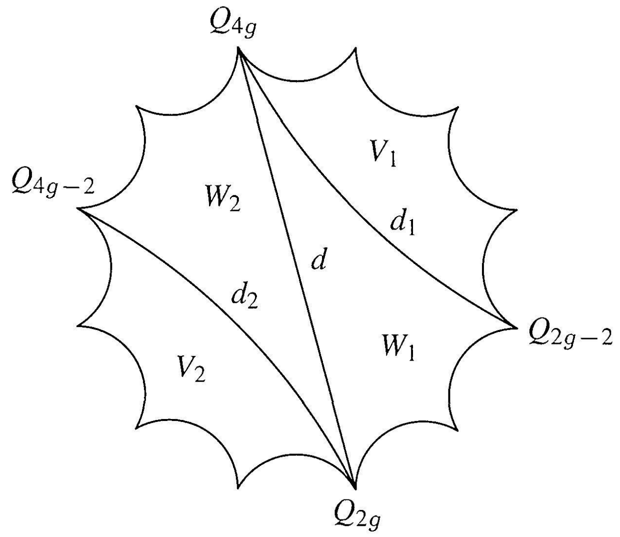

FIGURE 6

The partition of a canonical polygon $P(g)$ into two $(2g-1)$-gons and two quadrilaterals

Assume now that M has $2g-2$ simple closed geodesics $u_1, \dots, u_{2g-2}$ which all intersect in the same point Q and intersect in no other point. By Lemma 13 we then can find simple closed curves u_{2g-1} and u_{2g} such that

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon $P(g)$ with the usual notation. For $i = 1, \ldots, 4g$, let $\{Q_i\} = a_i \cap a_{i+1}$. In $P(g)$ let d_1 be the geodesic segment from Q_{4g} to Q_{2g-2}, d_2 the geodesic segment from Q_{2g} to Q_{4g-2}, and d the geodesic segment from Q_{2g} to Q_{4g}, compare Figure 6. Then $P(g) \setminus (d_1 \cup d_2 \cup d)$ has four connected components, two quadrilaterals W_j having d and d_j, $j = 1, 2$, among the sides and two $(2g-1)$-gons V_j having d_j among the sides, $j = 1, 2$. Since u_i, $i = 1, \ldots, 2g-2$, are simple closed geodesics, it follows that $\alpha_i = \alpha_{i+2g}$ for $i = 1, \ldots, 2g-3$. This implies that V_1 and V_2 are isometric and that d_1 and d_2 have the same length. Therefore, W_1 and W_2 are quadrilaterals with equal lengths of the four sides. Fix now W_1 and try to vary W_2 such that the lengths of the sides remain invariant and so that property (V) for canonical polygons holds. This is certainly the case if W_2 and W_1 are isometric. But then Corollary 8 implies that this is the unique possibility. Therefore, W_1 and W_2 must be isometric and hence $\alpha_i = \alpha_{i+2g}$ for all $i = 1, \ldots, 2g$, which proves (ii) $\Rightarrow$ (iii).

Now assume that (iii) holds. Let d be the geodesic segment from Q_{2g} to Q_{4g}. Then d separates $P(g)$ into two isometric $(2g+1)$-gons and the π-rotation around the centre C of d induces an isometry ϕ of M with $\phi^2 = id$. The fixed points of ϕ are C, the point Q corresponding to the vertices of $P(g)$ as well as the centres of the sides a_i, $i = 1, \ldots, 2g$. Therefore, ϕ is a hyperelliptic involution which proves (iii) $\Rightarrow$ (i). □

COROLLARY 15. *All closed hyperbolic surfaces of genus 2 are hyperelliptic.*

Proof. All closed hyperbolic surfaces have two simple closed geodesics which intersect in a unique point. The corollary follows by Theorem 14. □

DEFINITION. Let M_0 be a closed hyperbolic surface in T_g. For every $M \in T_g$ fix a homeomorphism ϕ_M, homotopic to the identity, from M_0 to M (ϕ_M exists since closed surfaces of the same genus are homeomorphic). Let u be a simple closed geodesic in M_0. Then, in the homotopy class of $\phi_M(u)$ there exists a unique simple closed geodesic which is denoted by $\Phi_M(u)$. The function

$$L(u)\colon T_g \to \mathbf{R}$$

which associates to M the length of $\Phi_M(u)$ is called a *geodesic length function*.

REMARK. It is well known that T_g can be parametrized by a finite number of geodesic length functions, see for example [12], [13] where it is shown that T_g can be parametrized by $6g-5$ geodesic length functions.

THEOREM 16. *The Teichmüller space T_g for $g=2$ can be parametrized by 7 (suitably chosen) geodesic length functions $L(u_1), \ldots, L(u_7)$, taken as homogeneous parameters (which means that $L(u_1)/L(u_7), \ldots, L(u_6)/L(u_7)$ gives a parametrization of T_2).*

Proof. Let $P(2)$ be a canonical polygon corresponding to a closed hyperbolic surface M_0 of genus 2. As usual let $Q_i = a_i \cap a_{i+1}$, $i = 1, \ldots, 8$, where the a_i are the sides of $P(2)$. Let b_i be the geodesic segment (in $P(2)$) between Q_i and Q_{i+4}, $i = 1, \ldots, 4$. By Corollary 15, M_0 is hyperelliptic, therefore (compare Theorem 14) b_i corresponds to a simple closed geodesic in M_0, denoted by B_i, $i = 1, \ldots, 4$. It also follows by Theorem 14 that a_i corresponds to a simple closed geodesic in M_0, denoted by A_i, $i = 1, \ldots, 4$.

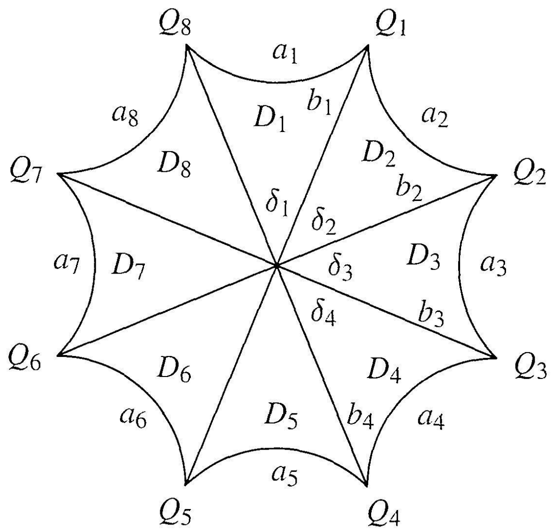

FIGURE 7

A triangulation of a canonical polygon $P(g)$ for $g=2$

I now prove that the 7 length functions, given by the simple closed geodesics A_i, $i = 1, 2, 3$, B_i, $i = 1, \ldots, 4$, taken as homogeneous parameters, give a parametrization of T_2. In order to do this, it is enough (by Theorem 11 and Corollary 12) to show that $P(2)$ is uniquely determined by the lengths of a_i, $i = 1, 2, 3$, b_i, $i = 1, \ldots, 4$, taken as homogeneous parameters (in the sequel I shall refer to these lengths calling them "the seven lengths"). This can be done analogously as in the proof of Theorem 11. The geodesic segments b_i, $i = 1, \ldots, 4$, intersect in a point C, the "centre" of $P(2)$, and they separate

$P(2)$ into 8 triangles D_j so that a_j is a side of D_j, $j = 1, \ldots, 8$, compare Figure 7. Since M is hyperelliptic, D_j and D_{j+4} are isometric, $j = 1, \ldots, 4$. Denote by δ_i the angle of D_i in the vertex C, $i = 1, \ldots, 4$. The seven lengths determine the triangles D_i, $i = 1, 2, 3$, as well as two sides and the angle δ_4 of D_4 by the condition

$$\Delta := \sum_{j=1}^{4} \delta_j = \pi \,, \tag{6}$$

so they determine also D_4. This shows that the seven lengths determine $P(2)$. Multiply the seven lengths by a positive real t and assume that the seven new lengths also determine a canonical polygon $P_t(2)$. If $t > 1$, then δ_i, $i = 1, 2, 3$, are smaller in $P_t(2)$ than in $P(2)$ by Lemma 9, therefore, by (6), δ_4 is larger in $P_t(2)$ than in $P(2)$. It follows by Lemma 7 that the sum of the two other angles of D_4 is smaller in $P_t(2)$ than in $P(2)$. Since all angles in D_i, $i = 1, 2, 3$, are smaller in $P_t(2)$ than in $P(2)$ by Lemma 9, it follows that

$$\sum_{i=1}^{4} \alpha_i$$

is smaller in $P_t(2)$ than in $P(2)$. But this contradicts condition (II) of canonical polygons. An analogous contradiction follows if $t < 1$ proving thus that $t = 1$ and therefore the theorem. $\square$

REMARK. Theorem 16 is new. It is well known that $6g-6$ length functions can never parametrize T_g so that the situation of Theorem 16 is the best we can expect. It is not known whether $6g-5$ geodesic length functions, *taken as homogeneous parameters*, can parametrize T_g for $g \geq 3$.

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