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## 5. TEICHMÜLLER SPACE

DEFINITION. The space  $\mathcal{P}(g)$  of canonical polygons contains all canonical polygons  $P(g)$  with the topology  $P_j(g) \rightarrow P(g)$  if and only if the lengths of all sides converge and all angles converge, more precisely, if and only if

$$a_i(P_j(g)) \rightarrow a_i(P(g)), \quad i = 1, \dots, 4g,$$

(where  $a_i(P_j(g))$  is the side  $a_i$  of  $P_j(g)$ ) and

$$\alpha_i(P_j(g)) \rightarrow \alpha_i(P(g)), \quad i = 1, \dots, 4g,$$

(where  $\alpha_i(P_j(g))$  is the angle  $\alpha_i$  of  $P_j(g)$ ).

REMARKS. (i) Note that two canonical polygons  $P(g)$  and  $P'(g)$  may be isometric, but represent different points in  $\mathcal{P}(g)$ . They represent the same point if and only if there is an isometry mapping the side  $a_i(P(g))$  to the side  $a_i(P'(g))$ ,  $i = 1, \dots, 4g$  (and not to the side  $a_j(P'(g))$ ,  $j \neq i$ ). One expresses this fact by saying that the sides of the canonical polygons are *marked*.

(ii) One may calculate the dimension of  $\mathcal{P}(g)$  in the following heuristic way (this argument is modeled after one given in [16]). A canonical polygon has  $4g$  vertices. Each vertex is determined in  $\mathbf{H}$  by two (real) parameters, this gives  $8g$  parameters. The dimension of the space of isometries of  $\mathbf{H}$  is 3 so we remain with  $8g - 3$  parameters. By condition (I) of a canonical polygon we have  $2g$  equalities and each of the conditions (II), (IV), (V) gives one equality. We remain with

$$8g - 3 - 2g - 3 = 6g - 6$$

parameters.

THEOREM 11.  $\mathcal{P}(g)$  is homeomorphic to  $\mathbf{R}^{6g-6}$ .

REMARK. The following proof is new. The theorem was first proved by Coldewey and Zieschang in an annex to [17], see also [18]. An (indirect) proof has also been given by Buser [2], compare the introduction.

*Proof.* (i) Let  $P(g)$  be a canonical polygon with sides  $a_i$  and angles  $\alpha_i$  between  $a_i$  and  $a_{i+1}$ ,  $i = 1, \dots, 4g$  (the indices are taken modulo  $4g$ ). Let  $\{Q_i\} = a_i \cap a_{i+1}$ ,  $i = 1, \dots, 4g$ . Denote by  $b_i$  the geodesic segment between

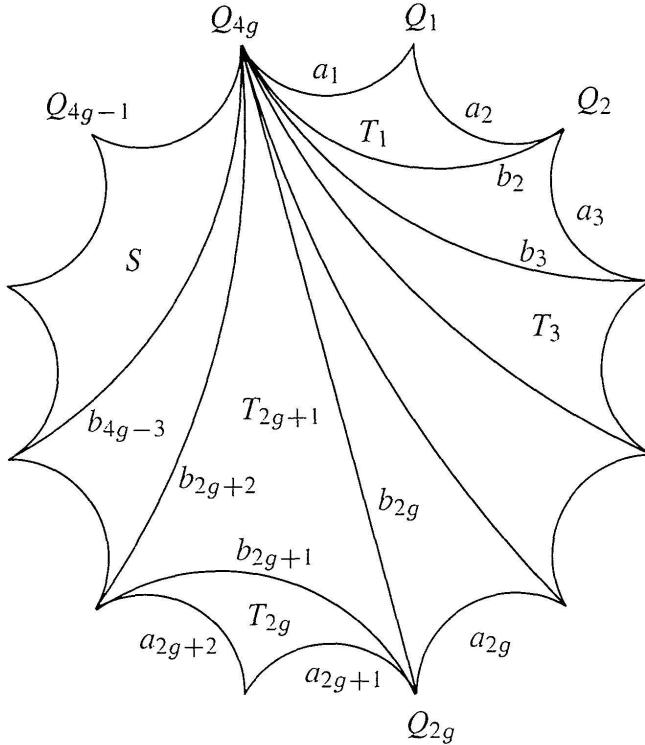


FIGURE 5  
The “triangulation” of a canonical polygon  $P(g)$

$Q_{4g}$  and  $Q_i$ ,  $i = 2, \dots, 4g - 3$ ,  $i \neq 2g + 1$ . Denote by  $b_{2g+1}$  the geodesic segment between  $Q_{2g}$  and  $Q_{2g+2}$ , compare Figure 5.

$P(g)$  is separated by the geodesic segments  $b_2, \dots, b_{4g-3}$  into one quadrilateral  $S$  and  $4g - 4$  triangles  $T_i$ ,  $i = 1, \dots, 4g - 4$ , with sides  $b_i, b_{i+1}, a_{i+1}$  for  $i = 2, \dots, 4g - 4$ ,  $i \neq 2g$ ,  $i \neq 2g + 1$ ; the triangle  $T_1$  has sides  $a_1, a_2, b_2$ , the triangle  $T_{2g}$  has sides  $a_{2g+1}, a_{2g+2}, b_{2g+1}$ , and the triangle  $T_{2g+1}$  has sides  $b_{2g}, b_{2g+1}, b_{2g+2}$  (note that  $T_{2g+1}$  is only defined if  $g > 2$ ).

A point  $x = (x_1, \dots, x_{6g-5}) \in \mathbf{R}^{6g-5}$  is called *admissible* if  $x_j > 0$ ,  $j = 1, \dots, 6g - 5$ , and if, putting

$$L(a_i) = L(a_{i+2g}) = x_i, \quad i = 1, \dots, 2g \quad (L = \text{length})$$

and

$$L(b_2) = L(b_{2g+1}) = x_{2g+1}$$

and

$$L(b_i) = x_{2g+i-1}, \quad i = 3, \dots, 2g; \quad L(b_i) = x_{2g+i-2}, \quad i = 2g + 2, \dots, 4g - 3,$$

the triangle inequalities hold for the triangles  $T_k$ ,  $k = 1, \dots, 4g - 4$ , and the “quadrilateral inequalities” hold for  $S$  (which means that the sum of the lengths of any three sides of  $S$  is greater than the length of the fourth side). Note that these are purely algebraic conditions on  $x \in \mathbf{R}^{6g-5}$ .

Let  $O$  be the subset of  $\mathbf{R}^{6g-5}$  of admissible points. Being the intersection of a finite number of open sets,  $O$  is open. Moreover,  $O$  is convex since  $O$  is the intersection of a finite number of convex sets, namely, if for example  $x_1 + x_2 > x_3$  and  $y_1 + y_2 > y_3$ , then

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) > \lambda x_3 + (1 - \lambda)y_3, \quad \forall \lambda \in [0, 1].$$

(ii) Let  $x \in O$ . Then we associate a formal polygon  $P(x)$  to  $x$  in the following way.  $P(x)$  is the formal union of the triangles  $T_k(x)$ ,  $k = 1, \dots, 4g - 4$ , and the quadrilateral  $S(x)$  in the same way as  $P(g)$ . Hereby, the triangles, as well as the lengths of the sides of  $S(x)$  are defined by the identifications described in part (i). The angles of the triangles are determined by their sides (by Theorem 6). The (formal) angles  $\alpha_i$  of  $P(x)$ ,  $i = 1, \dots, 4g$ , are defined as the sum of the angles of the corresponding triangles and (if  $i \in \{4g - 3, 4g - 2, 4g - 1, 4g\}$ ) of  $S(x)$ . Thereby, the angles of  $S(x)$  are defined by the conditions that  $S(x)$  is convex and that

$$\left| \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^g \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal, this minimum is denoted by  $\mathbf{m}(x)$ . By Corollary 10 the angles of  $S(x)$  are then determined and hence also the angles of  $P(x)$ . Note however that an angle  $\alpha_i$  of  $P(x)$  may be greater than  $2\pi$ , this is why  $P(x)$  is called a formal polygon with formally defined angles.

(iii) Let  $x \in O$ . Then  $tx$  (for  $t \in \mathbf{R}$ ,  $t > 0$ ) is also in  $O$  (since the triangle inequalities remain true). I claim that there exists a unique  $t_0 > 0$  (depending on  $x$ ) such that  $P(t_0x)$  is a canonical polygon. I first show uniqueness. Assume that  $\mathbf{m}(tx) > 0$  for  $P(tx)$ . This means that  $\mathbf{A}(tx) - \mathbf{B}(tx) \neq 0$  where

$$\mathbf{A}(tx) := \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} \quad \text{and} \quad \mathbf{B}(tx) := \sum_{i=1}^g \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}.$$

If  $\mathbf{A}(tx) - \mathbf{B}(tx) > 0$ , then an angle in  $S(tx)$  must be  $\pi$  and, by Corollary 8 and the minimality of  $\mathbf{m}(x)$ , this angle must appear in the sum  $\mathbf{B}(tx)$ . This implies that

$$(5) \quad \Sigma(tx) := \mathbf{A}(tx) + \mathbf{B}(tx) > 2\pi.$$

Of course, (5) also holds if  $\mathbf{A}(tx) - \mathbf{B}(tx) < 0$ . It follows that if  $P(t_0x)$  is a canonical polygon, then  $\mathbf{m}(t_0x) = 0$  (since  $\Sigma(t_0x) = 2\pi$  by the definition of canonical polygons). Now assume that  $P(t_0x)$  and  $P(t_1x)$  are canonical polygons with  $t_1 > t_0$ . By Lemma 9, all angles of the triangles  $T_k(t_1x)$

are smaller than the corresponding angles in  $T_k(t_0x)$ ,  $k = 1, \dots, 4g - 4$ . Moreover, by Corollary 10, at least two opposite angles in  $S(t_1x)$  are smaller than the corresponding angles in  $S(t_0x)$ . This implies that  $\mathbf{A}(t_1x) < \mathbf{A}(t_0x)$  or  $\mathbf{B}(t_1x) < \mathbf{B}(t_0x)$ . But since  $\mathbf{A}(t_1x) = \mathbf{B}(t_1x)$  and  $\mathbf{A}(t_0x) = \mathbf{B}(t_0x)$  ( $\mathbf{m}(t_0x) = \mathbf{m}(t_1x) = 0$ ), it follows that  $\Sigma(t_1x) < \Sigma(t_0x)$ , a contradiction. This proves uniqueness.

As for existence note that if  $t \rightarrow 0$ , then the volume of all triangles  $T_k$ ,  $k = 1, \dots, 4g - 4$ , and the volume of  $S$  tend to zero which implies by Theorem 3 that

$$\Sigma := \sum_{i=1}^{4g} \alpha_i \rightarrow (4g - 2)\pi.$$

On the other hand, for  $t \rightarrow \infty$ , all angles in the triangles  $T_k$ ,  $k = 1, \dots, 4g - 4$ , converge to zero by Lemma 9 and, by Corollary 10(ii), at least two opposite angles of  $S$  converge to zero. It follows by the condition that

$$\left| \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^g \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal that all angles of  $S$  converge to zero and hence  $\Sigma$  converge to zero. Therefore, there exists a  $t_0$  such that  $\Sigma(t_0x) = 2\pi$ . Now  $P(t_0x)$  is a canonical polygon. Namely, conditions (I), (II) and (IV) hold by construction. By the argument above, we further have  $\mathbf{m}(t_0x) = 0$  and condition (V) holds. Finally, condition (III) holds since all sides of the triangles of  $P(t_0x)$  have finite length and since conditions (II) and (V) hold.

(iv) We therefore have defined a projection from the open convex set  $O$  to the unit sphere in  $\mathbf{R}^{6g-5}$ . Since all operations are controlled by the formulas of Theorem 6, it is clear that this map is continuous and that the image is homeomorphic to  $\mathbf{R}^{6g-6}$  as well as homeomorphic to  $\mathcal{P}(g)$  since every canonical polygon is thereby obtained.  $\square$

**DEFINITION.** By Theorem 5 each of the canonical polygons in  $\mathcal{P}(g)$  defines a closed hyperbolic surface of genus  $g$ . The *Teichmüller space*  $T_g$  is the space of these hyperbolic surfaces with the topology induced from that of  $\mathcal{P}(g)$ .

**COROLLARY 12.**  $T_g$  is homeomorphic to  $\mathbf{R}^{6g-6}$ .  $\square$