

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS
Autor: SCHMUTZ SCHALLER, Paul
Kapitel: 5. Teichmüller space
DOI: <https://doi.org/10.5169/seals-64444>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

5. TEICHMÜLLER SPACE

DEFINITION. The space $\mathcal{P}(g)$ of canonical polygons contains all canonical polygons $P(g)$ with the topology $P_j(g) \rightarrow P(g)$ if and only if the lengths of all sides converge and all angles converge, more precisely, if and only if

$$a_i(P_j(g)) \rightarrow a_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $a_i(P_j(g))$ is the side a_i of $P_j(g)$) and

$$\alpha_i(P_j(g)) \rightarrow \alpha_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $\alpha_i(P_j(g))$ is the angle α_i of $P_j(g)$).

REMARKS. (i) Note that two canonical polygons $P(g)$ and $P'(g)$ may be isometric, but represent different points in $\mathcal{P}(g)$. They represent the same point if and only if there is an isometry mapping the side $a_i(P(g))$ to the side $a_i(P'(g))$, $i = 1, \dots, 4g$ (and not to the side $a_j(P'(g))$, $j \neq i$). One expresses this fact by saying that the sides of the canonical polygons are *marked*.

(ii) One may calculate the dimension of $\mathcal{P}(g)$ in the following heuristic way (this argument is modeled after one given in [16]). A canonical polygon has $4g$ vertices. Each vertex is determined in \mathbf{H} by two (real) parameters, this gives $8g$ parameters. The dimension of the space of isometries of \mathbf{H} is 3 so we remain with $8g - 3$ parameters. By condition (I) of a canonical polygon we have $2g$ equalities and each of the conditions (II), (IV), (V) gives one equality. We remain with

$$8g - 3 - 2g - 3 = 6g - 6$$

parameters.

THEOREM 11. $\mathcal{P}(g)$ is homeomorphic to \mathbf{R}^{6g-6} .

REMARK. The following proof is new. The theorem was first proved by Coldewey and Zieschang in an annex to [17], see also [18]. An (indirect) proof has also been given by Buser [2], compare the introduction.

Proof. (i) Let $P(g)$ be a canonical polygon with sides a_i and angles α_i between a_i and a_{i+1} , $i = 1, \dots, 4g$ (the indices are taken modulo $4g$). Let $\{Q_i\} = a_i \cap a_{i+1}$, $i = 1, \dots, 4g$. Denote by b_i the geodesic segment between

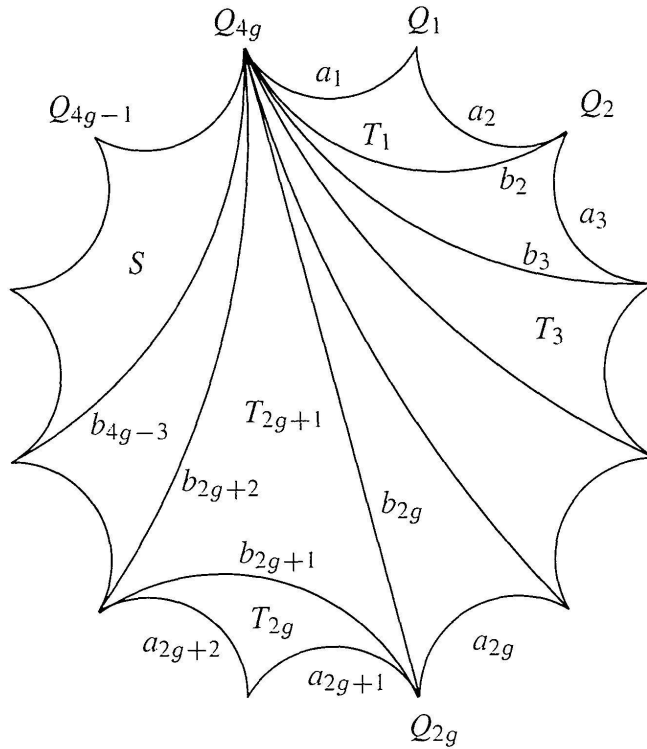


FIGURE 5

The “triangulation” of a canonical polygon $P(g)$

Q_{4g} and Q_i , $i = 2, \dots, 4g - 3$, $i \neq 2g + 1$. Denote by b_{2g+1} the geodesic segment between Q_{2g} and Q_{2g+2} , compare Figure 5.

$P(g)$ is separated by the geodesic segments b_2, \dots, b_{4g-3} into one quadrilateral S and $4g - 4$ triangles T_i , $i = 1, \dots, 4g - 4$, with sides b_i, b_{i+1}, a_{i+1} for $i = 2, \dots, 4g - 4$, $i \neq 2g$, $i \neq 2g + 1$; the triangle T_1 has sides a_1, a_2, b_2 , the triangle T_{2g} has sides $a_{2g+1}, a_{2g+2}, b_{2g+1}$, and the triangle T_{2g+1} has sides $b_{2g}, b_{2g+1}, b_{2g+2}$ (note that T_{2g+1} is only defined if $g > 2$).

A point $x = (x_1, \dots, x_{6g-5}) \in \mathbf{R}^{6g-5}$ is called *admissible* if $x_j > 0$, $j = 1, \dots, 6g - 5$, and if, putting

$$L(a_i) = L(a_{i+2g}) = x_i, \quad i = 1, \dots, 2g \quad (L = \text{length})$$

and

$$L(b_2) = L(b_{2g+1}) = x_{2g+1}$$

and

$$L(b_i) = x_{2g+i-1}, \quad i = 3, \dots, 2g; \quad L(b_i) = x_{2g+i-2}, \quad i = 2g + 2, \dots, 4g - 3,$$

the triangle inequalities hold for the triangles T_k , $k = 1, \dots, 4g - 4$, and the “quadrilateral inequalities” hold for S (which means that the sum of the lengths of any three sides of S is greater than the length of the fourth side). Note that these are purely algebraic conditions on $x \in \mathbf{R}^{6g-5}$.

Let O be the subset of \mathbf{R}^{6g-5} of admissible points. Being the intersection of a finite number of open sets, O is open. Moreover, O is convex since O is the intersection of a finite number of convex sets, namely, if for example $x_1 + x_2 > x_3$ and $y_1 + y_2 > y_3$, then

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) > \lambda x_3 + (1 - \lambda)y_3, \quad \forall \lambda \in [0, 1].$$

(ii) Let $x \in O$. Then we associate a formal polygon $P(x)$ to x in the following way. $P(x)$ is the formal union of the triangles $T_k(x)$, $k = 1, \dots, 4g - 4$, and the quadrilateral $S(x)$ in the same way as $P(g)$. Hereby, the triangles, as well as the lengths of the sides of $S(x)$ are defined by the identifications described in part (i). The angles of the triangles are determined by their sides (by Theorem 6). The (formal) angles α_i of $P(x)$, $i = 1, \dots, 4g$, are defined as the sum of the angles of the corresponding triangles and (if $i \in \{4g - 3, 4g - 2, 4g - 1, 4g\}$) of $S(x)$. Thereby, the angles of $S(x)$ are defined by the conditions that $S(x)$ is convex and that

$$\left| \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^g \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal, this minimum is denoted by $\mathbf{m}(x)$. By Corollary 10 the angles of $S(x)$ are then determined and hence also the angles of $P(x)$. Note however that an angle α_i of $P(x)$ may be greater than 2π , this is why $P(x)$ is called a formal polygon with formally defined angles.

(iii) Let $x \in O$. Then tx (for $t \in \mathbf{R}$, $t > 0$) is also in O (since the triangle inequalities remain true). I claim that there exists a unique $t_0 > 0$ (depending on x) such that $P(t_0x)$ is a canonical polygon. I first show uniqueness. Assume that $\mathbf{m}(tx) > 0$ for $P(tx)$. This means that $\mathbf{A}(tx) - \mathbf{B}(tx) \neq 0$ where

$$\mathbf{A}(tx) := \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} \quad \text{and} \quad \mathbf{B}(tx) := \sum_{i=1}^g \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}.$$

If $\mathbf{A}(tx) - \mathbf{B}(tx) > 0$, then an angle in $S(tx)$ must be π and, by Corollary 8 and the minimality of $\mathbf{m}(x)$, this angle must appear in the sum $\mathbf{B}(tx)$. This implies that

$$(5) \quad \Sigma(tx) := \mathbf{A}(tx) + \mathbf{B}(tx) > 2\pi.$$

Of course, (5) also holds if $\mathbf{A}(tx) - \mathbf{B}(tx) < 0$. It follows that if $P(t_0x)$ is a canonical polygon, then $\mathbf{m}(t_0x) = 0$ (since $\Sigma(t_0x) = 2\pi$ by the definition of canonical polygons). Now assume that $P(t_0x)$ and $P(t_1x)$ are canonical polygons with $t_1 > t_0$. By Lemma 9, all angles of the triangles $T_k(t_1x)$

are smaller than the corresponding angles in $T_k(t_0x)$, $k = 1, \dots, 4g - 4$. Moreover, by Corollary 10, at least two opposite angles in $S(t_1x)$ are smaller than the corresponding angles in $S(t_0x)$. This implies that $\mathbf{A}(t_1x) < \mathbf{A}(t_0x)$ or $\mathbf{B}(t_1x) < \mathbf{B}(t_0x)$. But since $\mathbf{A}(t_1x) = \mathbf{B}(t_1x)$ and $\mathbf{A}(t_0x) = \mathbf{B}(t_0x)$ ($\mathbf{m}(t_0x) = \mathbf{m}(t_1x) = 0$), it follows that $\Sigma(t_1x) < \Sigma(t_0x)$, a contradiction. This proves uniqueness.

As for existence note that if $t \rightarrow 0$, then the volume of all triangles T_k , $k = 1, \dots, 4g - 4$, and the volume of S tend to zero which implies by Theorem 3 that

$$\Sigma := \sum_{i=1}^{4g} \alpha_i \rightarrow (4g - 2)\pi.$$

On the other hand, for $t \rightarrow \infty$, all angles in the triangles T_k , $k = 1, \dots, 4g - 4$, converge to zero by Lemma 9 and, by Corollary 10(ii), at least two opposite angles of S converge to zero. It follows by the condition that

$$\left| \sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^g \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal that all angles of S converge to zero and hence Σ converge to zero. Therefore, there exists a t_0 such that $\Sigma(t_0x) = 2\pi$. Now $P(t_0x)$ is a canonical polygon. Namely, conditions (I), (II) and (IV) hold by construction. By the argument above, we further have $\mathbf{m}(t_0x) = 0$ and condition (V) holds. Finally, condition (III) holds since all sides of the triangles of $P(t_0x)$ have finite length and since conditions (II) and (V) hold.

(iv) We therefore have defined a projection from the open convex set O to the unit sphere in \mathbf{R}^{6g-5} . Since all operations are controlled by the formulas of Theorem 6, it is clear that this map is continuous and that the image is homeomorphic to \mathbf{R}^{6g-6} as well as homeomorphic to $\mathcal{P}(g)$ since every canonical polygon is thereby obtained. \square

DEFINITION. By Theorem 5 each of the canonical polygons in $\mathcal{P}(g)$ defines a closed hyperbolic surface of genus g . The *Teichmüller space* T_g is the space of these hyperbolic surfaces with the topology induced from that of $\mathcal{P}(g)$.

COROLLARY 12. T_g is homeomorphic to \mathbf{R}^{6g-6} . \square