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THEOREM 4. *Let Γ be a Fuchsian group without elliptic elements (an element $\gamma \in \text{PSL}(2, \mathbf{R})$ is elliptic if $|\text{tr}(\gamma)| < 2$ where tr is the trace). Then \mathbf{H}/Γ is a complete connected orientable Riemannian manifold of dimension 2 with a metric of constant curvature -1 .*

DEFINITION. *A hyperbolic surface is a connected orientable manifold $M = \mathbf{H}/\Gamma$ as in Theorem 4 (where Γ is a Fuchsian group without elliptic elements). M is called closed if M is compact and has no boundary.*

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DEFINITION (Compare Figure 2). Let $g \geq 2$ be an integer. A *canonical polygon* $P(g)$ is a polygon with $4g$ sides, denoted by a_1, \dots, a_{4g} , ordered clockwise, and angles α_i between a_i and a_{i+1} , $i = 1, \dots, 4g$ (indices are taken modulo $4g$), such that

- (I) a_i and a_{i+2g} have the same length, $i = 1, \dots, 2g$;
- (II) the sum of the angles of $P(g)$ is 2π ;
- (III) $0 < \alpha_i < \pi$, $i = 1, \dots, 4g$;
- (IV) $\alpha_1 = \alpha_{2g+1}$;
- (V) $\sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^g \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$.

I shall speak of condition (I) (or (II) or (III) or (IV) or (V)) referring to this definition.

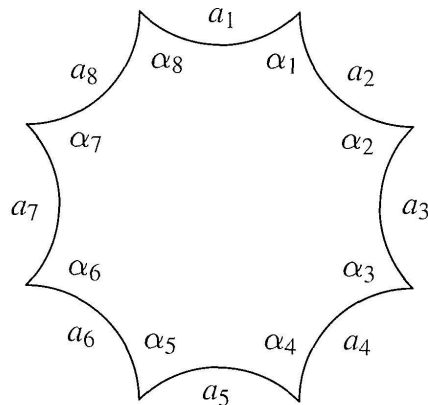


FIGURE 2
A canonical polygon $P(g)$ for $g = 2$

REMARKS. (i) Note that, by condition (II), both sides of the equation in condition (V) equal π .

(ii) The terminology *canonical* polygon is not standard, one finds different objects called canonical polygons in the literature (see for example in [15]).

DEFINITION. Let Γ be a Fuchsian group. A *fundamental domain* for Γ is a measurable subset D of \mathbf{H} such that

- (i) $\bigcup_{\gamma \in \Gamma} \gamma(D) = \mathbf{H}$, and
- (ii) $\text{int}(\bar{D}) \cap \text{int}(\gamma(\bar{D})) = \emptyset$ for $id \neq \gamma \in \Gamma$. Here, $\text{int}(S)$ is the *interior* of a set S and id is the unit matrix.

THEOREM 5 (Poincaré). A *canonical polygon* $P = P(g)$ is the *fundamental domain* of a Fuchsian group Γ and \mathbf{H}/Γ is a closed hyperbolic surface of genus g . The group Γ is generated by the $2g$ elements γ_i where γ_i is defined by the conditions $\gamma_i(P) \cap \text{int}(P) = \emptyset$ and $\gamma_i(a_i) = a_{i+2g}$ if i is odd and $\gamma_i(a_{i+2g}) = a_i$ if i is even, $i = 1, \dots, 2g$.

REMARKS. (i) For a proof see for example Poincaré [10], Siegel [15], Beardon [1], Iversen [5]. The theorem holds for much more general polygons. A general proof was first given by Maskit [9] and by de Rham [11].

(ii) Traditionally, the $2g$ generators γ_i of a Fuchsian group corresponding to a closed hyperbolic surface of genus g are chosen such that the relation

$$\prod_{i=1}^{2g} [\gamma_{2i-1}, \gamma_{2i}] = id$$

holds where

$$[\gamma_{2i-1}, \gamma_{2i}] = \gamma_{2i-1} \gamma_{2i} (\gamma_{2i-1})^{-1} (\gamma_{2i})^{-1}.$$

With the choice made here, the relation

$$\gamma_1 \gamma_2 \cdots \gamma_{2g} (\gamma_1)^{-1} (\gamma_2)^{-1} \cdots (\gamma_{2g})^{-1} = id$$

holds. Compare the introduction for the reasons for this choice.

(iii) Let $P(g)$ be a canonical polygon and $M = \mathbf{H}/\Gamma$ be the corresponding closed hyperbolic surface. Then the vertices of $P(g)$ correspond to a unique point Q in M and the side a_i (as well as a_{2g+i}) of $P(g)$ corresponds to a simple closed curve u_i in M , $i = 1, \dots, 2g$. These curves all intersect transversally in Q and intersect in no other point. Moreover, these curves are geodesic loops based in Q , this means that the curves may have an angle $\neq \pi$ in Q , but outside Q , they are geodesic. Further, condition (IV) and

condition (V) of canonical polygons are equivalent to the condition that u_1 and u_2 are simple closed geodesics in M .

4. TRIGONOMETRY

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

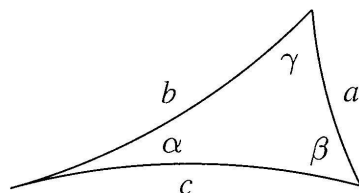


FIGURE 3

The notation for a triangle

THEOREM 6. Let T be a triangle with angles α, β, γ and sides of length a, b, c with the notation of Figure 3. Then

- (i) $\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$;
- (ii) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$;
- (iii) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$.

LEMMA 7. Let T be a triangle with the notation of Figure 3. Let T' be a triangle with sides of length a', b', c' and angles α', β', γ' . Let $a = a'$ and $b = b'$. Then

$$c' > c \iff \gamma' > \gamma \iff \alpha' + \beta' < \alpha + \beta .$$

Proof. The first equivalence is a consequence of Theorem 6 (ii).

Let Z be the centre of the side c and let u be the geodesic segment, of length $d/2$ say, between Z and the vertex C of T . The segment u separates T into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$