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canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space  $T_g$  for  $g = 2$  can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

## 2. HYPERBOLIC GEOMETRY AND FUCHSIAN GROUPS

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

DEFINITION. (i)  $\mathbf{H} = \{z = (x, y) \in \mathbf{C} : y > 0\}$  denotes the *upper halfplane*. The *hyperbolic metric* on  $\mathbf{H}$  is given by

$$dz = \frac{1}{y}(dz)_E$$

where  $(dz)_E$  is the standard Euclidean metric on  $\mathbf{C}$  and  $y$  is the imaginary part of  $z$ .

(ii) Define

$$\mathrm{SL}(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1; a, b, c, d \in \mathbf{R} \right\}$$

and

$$\mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\sim$$

with  $A \sim B$  if and only if  $A = \pm B$  for  $A, B \in \mathrm{SL}(2, \mathbf{R})$ . Let  $\gamma \in \mathrm{SL}(2, \mathbf{R})$ ,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the action of  $\gamma$  on  $\mathbf{H}$  is defined as

$$\gamma(z) = \frac{az + b}{cz + d}$$

for  $z \in \mathbf{H}$ .

**THEOREM 1.**  $\mathbf{H}$  is a complete Riemannian manifold of constant curvature  $-1$ . The geodesics in  $\mathbf{H}$  are either Euclidean semicircles which are orthogonal to the real axis or vertical half-lines.

**THEOREM 2.**

(i)  $\text{PSL}(2, \mathbf{R}) = \text{Isom}^+(\mathbf{H})$ , the group of orientation preserving isometries of  $\mathbf{H}$ .

(ii) Let  $u$  and  $v$  be geodesics in  $\mathbf{H}$ , let  $z$  be on  $u$  and  $z'$  on  $v$ . Then there exists  $\gamma \in \text{PSL}(2, \mathbf{R})$  with  $\gamma(u) = v$  and  $\gamma(z) = z'$ .

**DEFINITION.** For a measurable subset  $G \subset \mathbf{H}$  define the volume  $\text{vol}(G)$  as

$$\text{vol}(G) = \int_G \frac{dx dy}{y^2}.$$

**REMARK.** The volume is invariant under  $\gamma \in \text{SL}(2, \mathbf{R})$ .

**CONVENTIONS.** (i) Speaking of triangles, quadrilaterals and polygons always means that the sides are hyperbolic geodesic segments in  $\mathbf{H}$ .

(ii) Speaking of *angles* in triangles, quadrilaterals and polygons always means *interior angles*.

**THEOREM 3.** The volume of a polygon with angles  $\alpha_i$ ,  $i = 1, 2, \dots, m$ ,  $m \geq 3$ , is

$$(m - 2)\pi - \sum_{i=1}^m \alpha_i.$$

**DEFINITION.** A Fuchsian group  $\Gamma$  is a discrete subgroup of  $\text{PSL}(2, \mathbf{R})$  where discrete means that the identity matrix is not a cluster point in  $\Gamma$  with respect to the topology induced by the standard topology of  $\mathbf{R}^4$ .

**THEOREM 4.** *Let  $\Gamma$  be a Fuchsian group without elliptic elements (an element  $\gamma \in \text{PSL}(2, \mathbf{R})$  is elliptic if  $|\text{tr}(\gamma)| < 2$  where  $\text{tr}$  is the trace). Then  $\mathbf{H}/\Gamma$  is a complete connected orientable Riemannian manifold of dimension 2 with a metric of constant curvature  $-1$ .*

**DEFINITION.** *A hyperbolic surface is a connected orientable manifold  $M = \mathbf{H}/\Gamma$  as in Theorem 4 (where  $\Gamma$  is a Fuchsian group without elliptic elements).  $M$  is called closed if  $M$  is compact and has no boundary.*

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

**DEFINITION** (Compare Figure 2). Let  $g \geq 2$  be an integer. A *canonical polygon*  $P(g)$  is a polygon with  $4g$  sides, denoted by  $a_1, \dots, a_{4g}$ , ordered clockwise, and angles  $\alpha_i$  between  $a_i$  and  $a_{i+1}$ ,  $i = 1, \dots, 4g$  (indices are taken modulo  $4g$ ), such that

- (I)  $a_i$  and  $a_{i+2g}$  have the same length,  $i = 1, \dots, 2g$ ;
- (II) the sum of the angles of  $P(g)$  is  $2\pi$ ;
- (III)  $0 < \alpha_i < \pi$ ,  $i = 1, \dots, 4g$ ;
- (IV)  $\alpha_1 = \alpha_{2g+1}$ ;
- (V)  $\sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^g \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$ .

I shall speak of condition (I) (or (II) or (III) or (IV) or (V) ) referring to this definition.

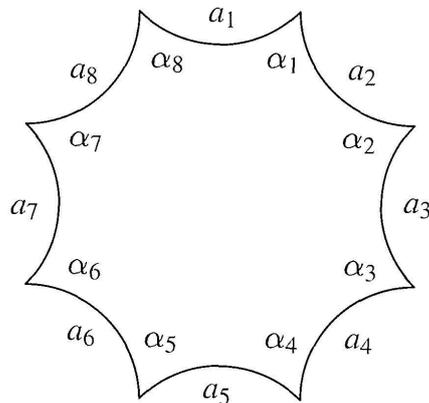


FIGURE 2  
A canonical polygon  $P(g)$  for  $g = 2$