Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF

FUCHSIAN GROUPS

Autor: SCHMUTZ SCHALLER, Paul

DOI: https://doi.org/10.5169/seals-64444

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 01.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

by Paul SCHMUTZ SCHALLER

1. Introduction

There are a number of ways to define the Teichmüller space of Riemann surfaces. In this paper I treat an approach which is less common than others. Let Γ be a Fuchsian group which uniformizes a closed Riemann surface of genus g. Then a fundamental domain for Γ is chosen in a canonical way, namely as a polygon with 4g sides such that opposite sides are identified. The Teichmüller space T_g of closed Riemann surfaces of genus g is then constructed by varying these polygons.

This construction of T_g by polygons was first done by Coldewey and Zieschang in an annex in [17], see also [18]; the construction includes the proof that T_g is homeomorphic to \mathbf{R}^{6g-6} . In [2], Buser gave a different, however indirect proof. Here, I propose a new construction and a new proof which is, in my eyes, easier and more transparent than the original one of Coldewey and Zieschang.

The main idea is the following. Let P(g) be a canonical polygon of 4g sides which is the fundamental domain of a Fuchsian group uniformizing a closed Riemann surfaces of genus g (the definition of P(g) will include some technical subtleties, to be discussed in Section 3). Then "triangulate" P(g) into 4g-4 triangles and one quadrilateral S. This can be done in such a way that these triangles are determined by 6g-5 positive real numbers (corresponding to the lengths of the sides of the triangles) with the condition that the different triangle inequalities hold. It turns out that these 6g-5 lengths, taken as homogeneous parameters, provide a parametrization of the Teichmüller space T_g . Since the set of reals for which the different triangle

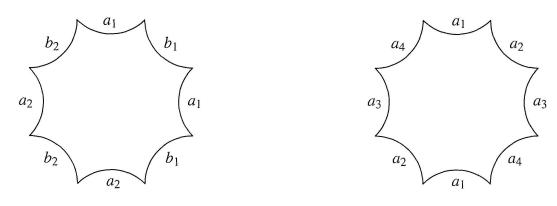


FIGURE 1

On the left hand side: usual identification
On the right hand side: identification chosen in this paper

inequalities hold is open and convex, this also proves that T_g is homeomorphic to \mathbf{R}^{6g-6} .

Let P be a polygon of 4g sides which is the fundamental domain for a Fuchsian group Γ uniformizing a closed Riemann surface M of genus g. This means that we can write

$$M = \mathbf{H}/\Gamma$$

where \mathbf{H} is the upper halfplane. Usually, P is chosen such that the identification of the sides of P is that of the polygon on the left hand side in Figure 1. The construction described above would equally work for these polygons. For the following reasons I prefer to choose the identification (compare the polygon on the right hand side of Figure 1) such that opposite sides are identified. First the sides of P correspond to simple (this means with no selfintersections) closed curves in M and if opposite sides are identified, then these simple closed curves intersect transversally (which is not the case with the usual identification). Secondly, the vertices of P correspond to a (unique) point Q in M; with the usual identification, Q is completely arbitrary while with the identification chosen here, there is a natural choice for Q in the case of hyperelliptic Riemann surfaces, namely, as a Weierstrass point. See Section 6 for details.

In this paper, I only treat the case of Fuchsian groups which uniformize closed Riemann surfaces. In a straightforward way, the construction and proof could be extended to all finitely generated Fuchsian groups. Note that concerning the original construction and proof in [17] (mentioned above) the corresponding generalization has been worked out by Coldewey in his thesis [3].

The paper is structured as follows. In Section 2 the basic definitions of hyperbolic geometry and Fuchsian groups are given. Section 3 defines the

canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space T_g for g=2 can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

2. HYPERBOLIC GEOMETRY AND FUCHSIAN GROUPS

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

DEFINITION. (i) $\mathbf{H} = \{z = (x, y) \in \mathbf{C} : y > 0\}$ denotes the *upper halfplane*. The *hyperbolic metric* on \mathbf{H} is given by

$$dz = \frac{1}{y}(dz)_E$$

where $(dz)_E$ is the standard Euclidean metric on \mathbb{C} and y is the imaginary part of z.

(ii) Define

$$SL(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1; a, b, c, d \in \mathbf{R} \right\}$$

and

$$PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\sim$$

with $A \sim B$ if and only if $A = \pm B$ for $A, B \in SL(2, \mathbf{R})$. Let $\gamma \in SL(2, \mathbf{R})$,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Then the action of γ on **H** is defined as

$$\gamma(z) = \frac{az+b}{cz+d}$$

for $z \in \mathbf{H}$.

THEOREM 1. **H** is a complete Riemannian manifold of constant curvature -1. The geodesics in **H** are either Euclidean semicircles which are orthogonal to the real axis or vertical half-lines.

THEOREM 2.

- (i) $PSL(2, \mathbf{R}) = Isom^+(\mathbf{H})$, the group of orientation preserving isometries of \mathbf{H} .
- (ii) Let u and v be geodesics in **H**, let z be on u and z' on v. Then there exists $\gamma \in PSL(2, \mathbf{R})$ with $\gamma(u) = v$ and $\gamma(z) = z'$.

DEFINITION. For a measurable subset $G \subset \mathbf{H}$ define the $volume\ vol(G)$ as

$$vol(G) = \int_G \frac{dx \, dy}{y^2} \, .$$

REMARK. The volume is invariant under $\gamma \in SL(2, \mathbf{R})$.

CONVENTIONS. (i) Speaking of triangles, quadrilaterals and polygons always means that the sides are hyperbolic geodesic segments in **H**.

(ii) Speaking of *angles* in triangles, quadrilaterals and polygons always means *interior angles*.

THEOREM 3. The volume of a polygon with angles α_i , $i=1,2,\ldots,m$, $m\geq 3$, is

$$(m-2)\pi-\sum_{i=1}^m\alpha_i.$$

DEFINITION. A Fuchsian group Γ is a discrete subgroup of PSL(2, \mathbf{R}) where discrete means that the identity matrix is not a cluster point in Γ with respect to the topology induced by the standard topology of \mathbf{R}^4 .

THEOREM 4. Let Γ be a Fuchsian group without elliptic elements (an element $\gamma \in PSL(2, \mathbf{R})$ is elliptic if $|tr(\gamma)| < 2$ where tr is the trace). Then \mathbf{H}/Γ is a complete connected orientable Riemannian manifold of dimension 2 with a metric of constant curvature -1.

DEFINITION. A hyperbolic surface is a connected orientable manifold $M = \mathbf{H}/\Gamma$ as in Theorem 4 (where Γ is a Fuchsian group without elliptic elements). M is called *closed* if M is compact and has no boundary.

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DEFINITION (Compare Figure 2). Let $g \ge 2$ be an integer. A canonical polygon P(g) is a polygon with 4g sides, denoted by a_1, \ldots, a_{4g} , ordered clockwise, and angles α_i between a_i and a_{i+1} , $i = 1, \ldots, 4g$ (indices are taken modulo 4g), such that

- (I) a_i and a_{i+2q} have the same length, $i = 1, \ldots, 2g$;
- (II) the sum of the angles of P(g) is 2π ;
- (III) $0 < \alpha_i < \pi, i = 1, \dots, 4g$;
- (IV) $\alpha_1 = \alpha_{2g+1}$;

(V)
$$\sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}.$$

I shall speak of condition (I) (or (II) or (III) or (IV) or (V)) referring to this definition.

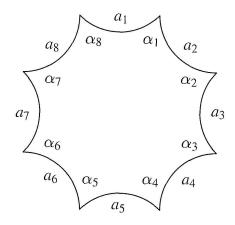


FIGURE 2 A canonical polygon P(g) for g = 2

REMARKS. (i) Note that, by condition (II), both sides of the equation in condition (V) equal π .

(ii) The terminology canonical polygon is not standard, one finds different objects called canonical polygons in the literature (see for example in [15]).

DEFINITION. Let Γ be a Fuchsian group. A fundamental domain for Γ is a measurable subset D of \mathbf{H} such that

- (i) $\bigcup_{\gamma \in \Gamma} \gamma(D) = \mathbf{H}$, and
- (ii) $int(\bar{D}) \cap int(\gamma(\bar{D})) = \emptyset$ for $id \neq \gamma \in \Gamma$. Here, int(S) is the *interior* of a set S and id is the unit matrix.

THEOREM 5 (Poincaré). A canonical polygon P = P(g) is the fundamental domain of a Fuchsian group Γ and \mathbf{H}/Γ is a closed hyperbolic surface of genus g. The group Γ is generated by the 2g elements γ_i where γ_i is defined by the conditions $\gamma_i(P) \cap int(P) = \emptyset$ and $\gamma_i(a_i) = a_{i+2g}$ if i is odd and $\gamma_i(a_{i+2g}) = a_i$ if i is even, $i = 1, \ldots, 2g$.

REMARKS. (i) For a proof see for example Poincaré [10], Siegel [15], Beardon [1], Iversen [5]. The theorem holds for much more general polygons. A general proof was first given by Maskit [9] and by de Rham [11].

(ii) Traditionally, the 2g generators γ_i of a Fuchsian group corresponding to a closed hyperbolic surface of genus g are chosen such that the relation

$$\prod_{i=1}^{2g} \left[\gamma_{2i-1}, \gamma_{2i} \right] = id$$

holds where

$$[\gamma_{2i-1}, \gamma_{2i}] = \gamma_{2i-1}\gamma_{2i}(\gamma_{2i-1})^{-1}(\gamma_{2i})^{-1}.$$

With the choice made here, the relation

$$\gamma_1 \gamma_2 \dots \gamma_{2q} (\gamma_1)^{-1} (\gamma_2)^{-1} \dots (\gamma_{2q})^{-1} = id$$

holds. Compare the introduction for the reasons for this choice.

(iii) Let P(g) be a canonical polygon and $M = \mathbf{H}/\Gamma$ be the corresponding closed hyperbolic surface. Then the vertices of P(g) correspond to a unique point Q in M and the side a_i (as well as a_{2g+i}) of P(g) corresponds to a simple closed curve u_i in M, $i=1,\ldots,2g$. These curves all intersect transversally in Q and intersect in no other point. Moreover, these curves are geodesic loops based in Q, this means that the curves may have an angle $\neq \pi$ in Q, but outside Q, they are geodesic. Further, condition (IV) and

condition (V) of canonical polygons are equivalent to the condition that u_1 and u_2 are simple closed geodesics in M.

4. Trigonometry

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

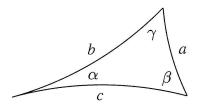


FIGURE 3

The notation for a triangle

THEOREM 6. Let T be a triangle with angles α, β, γ and sides of length a, b, c with the notation of Figure 3. Then

(i)
$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$
;

- (ii) $\cosh c = \cosh a \cosh b \sinh a \sinh b \cos \gamma$;
- (iii) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$.

LEMMA 7. Let T be a triangle with the notation of Figure 3. Let T' be a triangle with sides of length a', b', c' and angles α', β', γ' . Let a = a' and b = b'. Then

$$c' > c \iff \gamma' > \gamma \iff \alpha' + \beta' < \alpha + \beta$$
.

Proof. The first equivalence is a consequence of Theorem 6 (ii).

Let Z be the centre of the side c and let u be the geodesic segment, of length d/2 say, between Z and the vertex C of T. The segment u separates T into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$

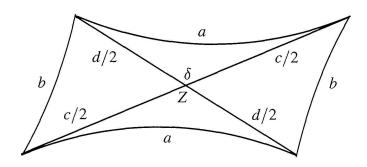


FIGURE 4

The triangle T (thick lines) is half of this quadrilateral

and

$$\cosh b = \cosh(c/2) \cosh(d/2) + \sinh(c/2) \sinh(d/2) \cos \delta$$

for an angle δ . This implies

(1)
$$\cosh a + \cosh b = 2 \cosh(c/2) \cosh(d/2).$$

Let \widetilde{T} be the triangle with sides of length a,b,d (compare Figure 4). Then the angles of \widetilde{T} are $\alpha+\beta,\gamma_1,\gamma_2$ with $\gamma=\gamma_1+\gamma_2$. Now if the length of c grows, then the length of d diminishes (by (1)), therefore, applying the first equivalence of the lemma to the triangle \widetilde{T} , the angle $\alpha+\beta$ diminishes and the second equivalence of the lemma follows. \square

COROLLARY 8. Let Q and Q' be two quadrilaterals with the same lengths of the four sides. Let $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ be the four angles in Q and Q', respectively, in the natural order (α and γ are opposite). Then

$$\alpha + \gamma > \alpha' + \gamma' \iff \beta + \delta < \beta' + \delta'$$
.

Proof. Clear by Lemma 7 (draw a diagonal in Q and in Q').

LEMMA 9. Let T be a triangle with the notation of Figure 3. Let T(t) be a triangle with sides of length ta, tb, tc and angles $\alpha_t, \beta_t, \gamma_t$.

- (i) If t > 1, then $\alpha_t < \alpha$, $\beta_t < \beta$, $\gamma_t < \gamma$.
- (ii) For $t \to \infty$, the three angles $\alpha_t, \beta_t, \gamma_t$ converge to zero.

Proof. (i) I prove $\gamma_t < \gamma$, the two other inequalities follow analogously. By Theorem 6(ii) it has to be shown that

(2)
$$\frac{\cosh ta \cosh tb - \cosh tc}{\sinh ta \sinh tb} - \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b} > 0.$$

By symmetry we can assume that $a \ge b$. Consider the left hand side of (2) as a function f = f(c) of c with fixed a, b, t. A calculation yields

(3)
$$f(a+b) = f(a-b) = 0.$$

Further, f'(c) = 0 implies

$$\frac{t \sinh tc}{\sinh c} = \frac{\sinh ta \sinh tb}{\sinh a \sinh b}$$

and by the convexity of the function sinh we conclude that f'(c) has only one zero. Since t > 1, it follows (by the definition of f) that

$$f(c) \to -\infty$$
 for $c \to \pm \infty$.

Therefore, by (3), f(c) > 0 for a - b < c < a + b, which is the triangle inequality, and $\gamma_t < \gamma$ follows.

(ii) Assume without restriction that $a \leq b \leq c$. It then follows by Theorem 6(i) that $\alpha \leq \beta \leq \gamma$. This implies by Theorem 6(ii) that α_t and β_t converge to zero for $t \to \infty$. We compare the triangle T(t) with the triangle T'(t) which has two sides of length t(a+b)/2 and one side of length tc. Denote by γ_t' the angle in T'(t) which is opposite to the side of length tc. By a similar (but easier) argument as in part (i) it follows that $\gamma_t' \geq \gamma_t$ for all $t \geq 1$. It is therefore sufficient to prove

(4)
$$\gamma_t' \to 0$$
, for $t \to \infty$.

By Theorem 6(i) we have

$$\sin\frac{\gamma_t'}{2} = \frac{\sinh(tc/2)}{\sinh(t(a+b)/2)}.$$

This implies (4) since c/2 < (a+b)/2 (by the triangle inequality).

COROLLARY 10. Let Q be a quadrilateral with sides of length a, b, c, d and angles $\alpha, \beta, \gamma, \delta$ (so that a and c are opposite sides and α and γ are opposite angles). Let Q(t) be a quadrilateral with sides of length ta, tb, tc, td and angles $\alpha_t, \beta_t, \gamma_t, \delta_t$ (the notation is analogous to that of Q).

- (i) If t > 1, then at least two opposite angles are smaller in Q(t) than in Q.
- (ii) For every $\epsilon > 0$, there exists a real $T(\epsilon)$ such that, for every $t > T(\epsilon)$, $\alpha_t + \gamma_t < \epsilon$ or $\beta_t + \delta_t < \epsilon$.

Proof. Let e be the length of a diagonal of Q. Construct the quadrilateral Q'(t) with a diagonal of length te and sides of length ta, tb, tc, td. By Lemma 9 all four angles of Q'(t) are smaller than the corresponding angles in Q and moreover converge to zero if $t \to \infty$. The corollary now follows by Corollary 8. \square

5. TEICHMÜLLER SPACE

DEFINITION. The space $\mathcal{P}(g)$ of canonical polygons contains all canonical polygons P(g) with the topology $P_j(g) \to P(g)$ if and only if the lengths of all sides converge and all angles converge, more precisely, if and only if

$$a_i(P_i(g)) \rightarrow a_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $a_i(P_i(g))$ is the side a_i of $P_i(g)$) and

$$\alpha_i(P_j(g)) \to \alpha_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $\alpha_i(P_j(g))$ is the angle α_i of $P_j(g)$).

REMARKS. (i) Note that two canonical polygons P(g) and P'(g) may be isometric, but represent different points in $\mathcal{P}(g)$. They represent the same point if and only if there is an isometry mapping the side $a_i(P(g))$ to the side $a_i(P'(g))$, $i = 1, \ldots, 4g$ (and not to the side $a_j(P'(g))$, $j \neq i$). One expresses this fact by saying that the sides of the canonical polygons are *marked*.

(ii) One may calculate the dimension of $\mathcal{P}(g)$ in the following heuristic way (this argument is modeled after one given in [16]). A canonical polygon has 4g vertices. Each vertex is determined in \mathbf{H} by two (real) parameters, this gives 8g parameters. The dimension of the space of isometries of \mathbf{H} is 3 so we remain with 8g-3 parameters. By condition (I) of a canonical polygon we have 2g equalities and each of the conditions (II), (IV), (V) gives one equality. We remain with

$$8g - 3 - 2g - 3 = 6g - 6$$

parameters.

THEOREM 11. $\mathcal{P}(g)$ is homeomorphic to \mathbf{R}^{6g-6} .

REMARK. The following proof is new. The theorem was first proved by Coldewey and Zieschang in an annex to [17], see also [18]. An (indirect) proof has also been given by Buser [2], compare the introduction.

Proof. (i) Let P(g) be a canonical polygon with sides a_i and angles α_i between a_i and a_{i+1} , i = 1, ..., 4g (the indices are taken modulo 4g). Let $\{Q_i\} = a_i \cap a_{i+1}, i = 1, ..., 4g$. Denote by b_i the geodesic segment between

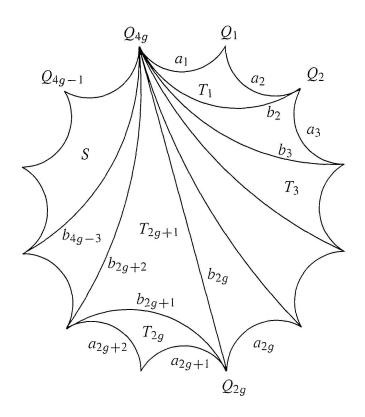


FIGURE 5 The "triangulation" of a canonical polygon P(g)

 Q_{4g} and Q_i , $i=2,\ldots,4g-3$, $i\neq 2g+1$. Denote by b_{2g+1} the geodesic segment between Q_{2g} and Q_{2g+2} , compare Figure 5.

P(g) is separated by the geodesic segments b_2, \ldots, b_{4g-3} into one quadrilateral S and 4g-4 triangles T_i , $i=1,\ldots,4g-4$, with sides b_i,b_{i+1},a_{i+1} for $i=2,\ldots,4g-4$, $i\neq 2g$, $i\neq 2g+1$; the triangle T_1 has sides a_1,a_2,b_2 , the triangle T_{2g} has sides $a_{2g+1},a_{2g+2},b_{2g+1}$, and the triangle T_{2g+1} has sides b_{2g},b_{2g+1},b_{2g+2} (note that T_{2g+1} is only defined if g>2).

A point $x = (x_1, \dots, x_{6g-5}) \in \mathbf{R}^{6g-5}$ is called *admissible* if $x_j > 0$, $j = 1, \dots, 6g-5$, and if, putting

$$L(a_i) = L(a_{i+2a}) = x_i, \quad i = 1, ..., 2q \quad (L = \text{length})$$

and

$$L(b_2) = L(b_{2g+1}) = x_{2g+1}$$

and

$$L(b_i) = x_{2g+i-1}, i = 3, ..., 2g; L(b_i) = x_{2g+i-2}, i = 2g+2, ..., 4g-3,$$

the triangle inequalities hold for the triangles T_k , k = 1, ..., 4g - 4, and the "quadrilateral inequalities" hold for S (which means that the sum of the lengths of any three sides of S is greater than the length of the fourth side). Note that these are purely algebraic conditions on $x \in \mathbb{R}^{6g-5}$.

Let O be the subset of \mathbb{R}^{6g-5} of admissible points. Being the intersection of a finite number of open sets, O is open. Moreover, O is convex since O is the intersection of a finite number of convex sets, namely, if for example $x_1 + x_2 > x_3$ and $y_1 + y_2 > y_3$, then

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) > \lambda x_3 + (1 - \lambda)(y_3), \ \forall \lambda \in [0, 1].$$

(ii) Let $x \in O$. Then we associate a formal polygon P(x) to x in the following way. P(x) is the formal union of the triangles $T_k(x)$, $k = 1, \ldots, 4g - 4$, and the quadrilateral S(x) in the same way as P(g). Hereby, the triangles, as well as the lengths of the sides of S(x) are defined by the identifications described in part (i). The angles of the triangles are determined by their sides (by Theorem 6). The (formal) angles α_i of P(x), $i = 1, \ldots, 4g$, are defined as the sum of the angles of the corresponding triangles and (if $i \in \{4g - 3, 4g - 2, 4g - 1, 4g\}$) of S(x). Thereby, the angles of S(x) are defined by the conditions that S(x) is convex and that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal, this minimum is denoted by $\mathbf{m}(x)$. By Corollary 10 the angles of S(x) are then determined and hence also the angles of P(x). Note however that an angle α_i of P(x) may be greater than 2π , this is why P(x) is called a formal polygon with formally defined angles.

(iii) Let $x \in O$. Then tx (for $t \in \mathbf{R}$, t > 0) is also in O (since the triangle inequalities remain true). I claim that there exists a unique $t_0 > 0$ (depending on x) such that $P(t_0x)$ is a canonical polygon. I first show uniqueness. Assume that $\mathbf{m}(tx) > 0$ for P(tx). This means that $\mathbf{A}(tx) - \mathbf{B}(tx) \neq 0$ where

$$\mathbf{A}(tx) := \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i}$$
 and $\mathbf{B}(tx) := \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$.

If $\mathbf{A}(tx) - \mathbf{B}(tx) > 0$, then an angle in S(tx) must be π and, by Corollary 8 and the minimality of $\mathbf{m}(x)$, this angle must appear in the sum $\mathbf{B}(tx)$. This implies that

(5)
$$\Sigma(tx) := \mathbf{A}(tx) + \mathbf{B}(tx) > 2\pi.$$

Of course, (5) also holds if $\mathbf{A}(tx) - \mathbf{B}(tx) < 0$. It follows that if $P(t_0x)$ is a canonical polygon, then $\mathbf{m}(t_0x) = 0$ (since $\Sigma(t_0x) = 2\pi$ by the definition of canonical polygons). Now assume that $P(t_0x)$ and $P(t_1x)$ are canonical polygons with $t_1 > t_0$. By Lemma 9, all angles of the triangles $T_k(t_1x)$

are smaller than the corresponding angles in $T_k(t_0x)$, $k=1,\ldots,4g-4$. Moreover, by Corollary 10, at least two opposite angles in $S(t_1x)$ are smaller than the corresponding angles in $S(t_0x)$. This implies that $\mathbf{A}(t_1x) < \mathbf{A}(t_0x)$ or $\mathbf{B}(t_1x) < \mathbf{B}(t_0x)$. But since $\mathbf{A}(t_1x) = \mathbf{B}(t_1x)$ and $\mathbf{A}(t_0x) = \mathbf{B}(t_0x)$ ($\mathbf{m}(t_0x) = \mathbf{m}(t_1x) = 0$), it follows that $\Sigma(t_1x) < \Sigma(t_0x)$, a contradiction. This proves uniqueness.

As for existence note that if $t \to 0$, then the volume of all triangles T_k , $k = 1, \ldots, 4g - 4$, and the volume of S tend to zero which implies by Theorem 3 that

$$\Sigma := \sum_{i=1}^{4g} \alpha_i \to (4g-2)\pi.$$

On the other hand, for $t \to \infty$, all angles in the triangles T_k , $k = 1, \ldots, 4g-4$, converge to zero by Lemma 9 and, by Corollary 10(ii), at least two opposite angles of S converge to zero. It follows by the condition that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal that all angles of S converge to zero and hence Σ converge to zero. Therefore, there exists a t_0 such that $\Sigma(t_0x)=2\pi$. Now $P(t_0x)$ is a canonical polygon. Namely, conditions (I), (II) and (IV) hold by construction. By the argument above, we further have $\mathbf{m}(t_0x)=0$ and condition (V) holds. Finally, condition (III) holds since all sides of the triangles of $P(t_0x)$ have finite length and since conditions (II) and (V) hold.

(iv) We therefore have defined a projection from the open convex set O to the unit sphere in \mathbb{R}^{6g-5} . Since all operations are controlled by the formulas of Theorem 6, it is clear that this map is continuous and that the image is homeomorphic to \mathbb{R}^{6g-6} as well as homeomorphic to $\mathcal{P}(g)$ since every canonical polygon is thereby obtained.

DEFINITION. By Theorem 5 each of the canonical polygons in $\mathcal{P}(g)$ defines a closed hyperbolic surface of genus g. The *Teichmüller space* T_g is the space of these hyperbolic surfaces with the topology induced from that of $\mathcal{P}(g)$.

COROLLARY 12. T_g is homeomorphic to \mathbf{R}^{6g-6} .

6. APPLICATIONS

LEMMA 13. Let M be a closed hyperbolic surface of genus g which has 2g-2 simple closed geodesics u_1, \ldots, u_{2g-2} which all intersect in the same point Q and intersect in no other point. Then M has simple closed curves u_{2g-1} and u_{2g} , passing through Q, such that the curves u_i intersect in no other point than Q, $i=1,\ldots,2g$. Moreover, u_{2g-1} and u_g can be chosen such that

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon P(g).

Proof. Cut M along u_1 , the result is a hyperbolic surface M_1 with boundary and genus g-1, the boundary consists of two simple closed geodesics v_1 and w_1 . Cut M_1 along u_2 , the result is a hyperbolic surface M_2 with one boundary component v_2 and genus g-1. Now cut M along all 2g-2 simple closed geodesics u_1, \ldots, u_{2g-2} . By induction, the result is a hyperbolic surface M_{2q-2} with one boundary component v and genus 1. More precisely, the boundary v is piecewise geodesic with 4g-4 pieces and we may assume that the notation is chosen such that these pieces appear on v in the order (the pieces are called like the corresponding closed curves) $u_1, u_2, \dots, u_{2g-2}, u_1, u_2, \dots, u_{2g-2}$ (note that closed geodesics intersect transversally). Denote by S and S' the two copies of Q on v between u_1 and u_{2g-2} . Let u_{2g-1} be a simple geodesic in M_{2g-2} which joins S and S' such that u_{2g-1} is not homotopic to a part of v. Cut M_{2g-2} along u_{2g-1} . The result is a hyperbolic surface M_{2q-1} of genus zero with two boundary components w and w' which both consist of 2g-1 geodesic pieces in the order $u_1, u_2, \ldots, u_{2g-2}, u_{2g-1}$. Denote by R and R' the copies of Q between u_1 and u_{2g-1} on w and w', respectively. Let u_{2g} be a simple geodesic in M_{2q-1} which joins R and R', u_{2q} can be chosen such that when we cut M_{2q-1} along u_{2q} , then we obtain the interior of a canonical polygon as desired.

DEFINITION. A hyperelliptic surface is a closed hyperbolic surface of genus g which has an isometry ϕ with $\phi^2=id$ and with exactly 2g+2 fixed points.

In [14], the equivalence of (i) and (ii) of the following theorem was first proved. With the approach chosen here, we can give a third equivalence and

a different proof.

Theorem 14. Let M be a closed hyperbolic surface M of genus g. Then the following conditions are equivalent.

- (i) M is hyperelliptic.
- (ii) M has a set of at least 2g-2 simple closed geodesics which all intersect in the same point and intersect in no other point.
- (iii) M has a corresponding canonical polygon with equal opposite angles $(\alpha_i = \alpha_{2g+i}, i = 1, ..., 2g)$.

Proof. I shall prove (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

Let M be hyperelliptic. Let R_i , $i=1,\ldots,2g+2$, be the fixed points of a hyperelliptic involution ϕ . Let c_1 be a simple geodesic segment from R_1 to R_2 . Then $c_1 \cup \phi(c_1)$ is a simple closed geodesic u_1 since $\phi^2 = id$. It also follows that on u_1 , there are only two fixed points of ϕ and that $M_1 = M \setminus u_1$ is connected. Therefore, we can choose a simple geodesic segment c_2 from R_1 to R_3 which intersects u_1 only in R_1 . By the same argument as above, $c_2 \cup \phi(c_2)$ is a simple closed geodesic, $M_2 = M \setminus (u_1 \cup u_2)$ is connected and on $u_1 \cup u_2$, there are only three fixed points of ϕ . Continuing this construction we can find simple closed geodesics u_1, \ldots, u_{2g-2} which all intersect in R_1 and in no other point. This proves (i) \Rightarrow (ii).

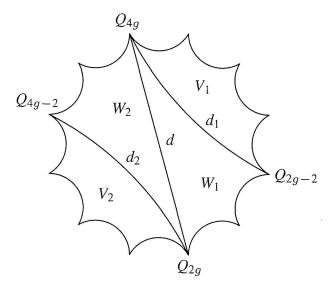


FIGURE 6

The partition of a canonical polygon P(g) into two (2g-1)-gons and two quadrilaterals

Assume now that M has 2g-2 simple closed geodesics u_1, \ldots, u_{2g-2} which all intersect in the same point Q and intersect in no other point. By Lemma 13 we then can find simple closed curves u_{2g-1} and u_{2g} such that

$$M\setminus \bigcup_{i=1}^{2g}u_i$$

is the interior of a canonical polygon P(g) with the usual notation. For $i=1,\ldots,4g$, let $\{Q_i\}=a_i\cap a_{i+1}$. In P(g) let d_1 be the geodesic segment from Q_{4g} to Q_{2g-2} , d_2 the geodesic segment from Q_{2g} to Q_{4g-2} , and d the geodesic segment from Q_{2g} to Q_{4g} , compare Figure 6. Then $P(g)\setminus (d_1\cup d_2\cup d)$ has four connected components, two quadrilaterals W_j having d and d_j , j=1,2, among the sides and two (2g-1)-gons V_j having d_j among the sides, j=1,2. Since u_i , $i=1,\ldots,2g-2$, are simple closed geodesics, it follows that $\alpha_i=\alpha_{i+2g}$ for $i=1,\ldots,2g-3$. This implies that V_1 and V_2 are isometric and that d_1 and d_2 have the same length. Therefore, W_1 and W_2 are quadrilaterals with equal lengths of the four sides. Fix now W_1 and try to vary W_2 such that the lengths of the sides remain invariant and so that property (V) for canonical polygons holds. This is certainly the case if W_2 and W_1 are isometric. But then Corollary 8 implies that this is the unique possibility. Therefore, W_1 and W_2 must be isometric and hence $\alpha_i=\alpha_{i+2g}$ for all $i=1,\ldots,2g$, which proves (ii) \Rightarrow (iii).

Now assume that (iii) holds. Let d be the geodesic segment from Q_{2g} to Q_{4g} . Then d separates P(g) into two isometric (2g+1)-gons and the π -rotation around the centre C of d induces an isometry ϕ of M with $\phi^2 = id$. The fixed points of ϕ are C, the point Q corresponding to the vertices of P(g) as well as the centres of the sides a_i , $i=1,\ldots,2g$. Therefore, ϕ is a hyperelliptic involution which proves (iii) \Rightarrow (i). \square

COROLLARY 15. All closed hyperbolic surfaces of genus 2 are hyperelliptic.

Proof. All closed hyperbolic surfaces have two simple closed geodesics which intersect in a unique point. The corollary follows by Theorem 14. \Box

DEFINITION. Let M_0 be a closed hyperbolic surface in T_g . For every $M \in T_g$ fix a homeomorphism ϕ_M , homotopic to the identity, from M_0 to M (ϕ_M exists since closed surfaces of the same genus are homeomorphic). Let u be a simple closed geodesic in M_0 . Then, in the homotopy class of $\phi_M(u)$ there exists a unique simple closed geodesic which is denoted by $\Phi_M(u)$. The function

$$L(u)\colon T_g\to \mathbf{R}$$

which associates to M the length of $\Phi_M(u)$ is called a geodesic length function.

REMARK. It is well known that T_g can be parametrized by a finite number of geodesic length functions, see for example [12], [13] where it is shown that T_g can be parametrized by 6g-5 geodesic length functions.

THEOREM 16. The Teichmüller space T_g for g=2 can be parametrized by 7 (suitably chosen) geodesic length functions $L(u_1), \ldots, L(u_7)$, taken as homogeneous parameters (which means that $L(u_1)/L(u_7), \ldots, L(u_6)/L(u_7)$ gives a parametrization of T_2).

Proof. Let P(2) be a canonical polygon corresponding to a closed hyperbolic surface M_0 of genus 2. As usual let $Q_i = a_i \cap a_{i+1}$, $i = 1, \ldots, 8$, where the a_i are the sides of P(2). Let b_i be the geodesic segment (in P(2)) between Q_i and Q_{i+4} , $i = 1, \ldots, 4$. By Corollary 15, M_0 is hyperelliptic, therefore (compare Theorem 14) b_i corresponds to a simple closed geodesic in M_0 , denoted by B_i , $i = 1, \ldots, 4$. It also follows by Theorem 14 that a_i corresponds to a simple closed geodesic in M_0 , denoted by A_i , $i = 1, \ldots, 4$.

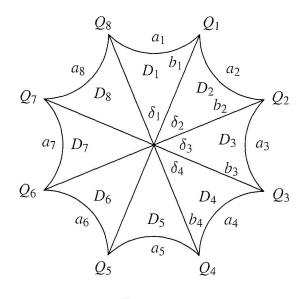


FIGURE 7

A triangulation of a canonical polygon P(g) for g=2

I now prove that the 7 length functions, given by the simple closed geodesics A_i , i=1,2,3, B_i , $i=1,\ldots,4$, taken as homogeneous parameters, give a parametrization of T_2 . In order to do this, it is enough (by Theorem 11 and Corollary 12) to show that P(2) is uniquely determined by the lengths of a_i , i=1,2,3, b_i , $i=1,\ldots,4$, taken as homogeneous parameters (in the sequel I shall refer to these lengths calling them "the seven lengths"). This can be done analogously as in the proof of Theorem 11. The geodesic segments b_i , $i=1,\ldots,4$, intersect in a point C, the "centre" of P(2), and they separate

P(2) into 8 triangles D_j so that a_j is a side of D_j , $j=1,\ldots,8$, compare Figure 7. Since M is hyperelliptic, D_j and D_{j+4} are isometric, $j=1,\ldots,4$. Denote by δ_i the angle of D_i in the vertex C, $i=1,\ldots,4$. The seven lengths determine the triangles D_i , i=1,2,3, as well as two sides and the angle δ_4 of D_4 by the condition

(6)
$$\Delta := \sum_{j=1}^{4} \delta_j = \pi ,$$

so they determine also D_4 . This shows that the seven lengths determine P(2). Multiply the seven lengths by a positive real t and assume that the seven new lengths also determine a canonical polygon $P_t(2)$. If t > 1, then δ_i , i = 1, 2, 3, are smaller in $P_t(2)$ than in P(2) by Lemma 9, therefore, by (6), δ_4 is larger in $P_t(2)$ than in P(2). It follows by Lemma 7 that the sum of the two other angles of D_4 is smaller in $P_t(2)$ than in P(2). Since all angles in D_i , i = 1, 2, 3, are smaller in $P_t(2)$ than in P(2) by Lemma 9, it follows that

$$\sum_{i=1}^{4} \alpha_i$$

is smaller in $P_t(2)$ than in P(2). But this contradicts condition (II) of canonical polygons. An analogous contradiction follows if t < 1 proving thus that t = 1 and therefore the theorem.

REMARK. Theorem 16 is new. It is well known that 6g-6 length functions can never parametrize T_g so that the situation of Theorem 16 is the best we can expect. It is not known whether 6g-5 geodesic length functions, taken as homogeneous parameters, can parametrize T_g for $g \ge 3$.

REFERENCES

- [1] BEARDON, A. F. The Geometry of Discrete Groups. Springer, 1983.
- [2] BUSER, P. Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser, Boston, 1992.
- [3] COLDEWEY, H.-D. Kanonische Polygone endlich erzeugter Fuchsscher Gruppen. Dissertation, Bochum, 1971.
- [4] FORD, L. Automorphic Functions. Chelsea, New York, 1929.
- [5] IVERSEN, B. Hyperbolic Geometry. Cambridge University Press, 1992.
- [6] JOST, J. Compact Riemann Surfaces. Springer, 1997.
- [7] KATOK, S. Fuchsian Groups. The University of Chicago Press, 1992.

- [8] LEHNER, J. Discontinuous groups and automorphic functions. *Math. Surveys, No. VIII*, AMS Providence, 1964.
- [9] MASKIT, B. On Poincaré's theorem for fundamental polygons. *Advances in math.* 7 (1971), 219–230.
- [10] POINCARÉ, H. Théorie des groupes fuchsiens. Acta math. 1 (1882), 1-62.
- [11] DE RHAM, G. Sur les polygones générateurs de groupes fuchsiens. L'Enseignement math. (2) 17 (1971), 49-61.
- [12] SCHMUTZ, P. Une paramétrisation de l'espace de Teichmüller de genre g donnée par 6g 5 géodésiques explicites. Sém. théorie spectrale et géométrie, Chambéry-Grenoble (1991–1992), 59–64.
- [13] Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen. *Comment. Math. Helv.* 68 (1993), 278–288.
- [14] SCHMUTZ SCHALLER, P. Geometric characterization of hyperelliptic Riemann surfaces. *Ann. Acad. Sci. Fenn. Math.* (to appear).
- [15] SIEGEL, C. L. Topics in Complex Function Theory. Vol. II. Wiley Interscience, 1969.
- [16] THURSTON, W. P. *Three-dimensional Geometry and Topology.* Vol. I. Princeton University Press, 1997.
- [17] ZIESCHANG, H., E. VOGT and H.-D. COLDEWEY. Flächen und ebene diskontinuierliche Gruppen. Springer LNM 122, 1970.
- [18] ZIESCHANG, H., E. VOGT and H.-D. COLDEWEY. Surfaces and Planar Discontinuous Groups. Springer LNM 835, 1980.

(Reçu le 6 janvier 1998; version révisée reçue le 26 octobre 1998)

Paul Schmutz Schaller

Institut de mathématiques Université de Neuchâtel Rue Emile-Argand 11 CH-2007 Neuchâtel Switzerland

e-mail: Paul.Schmutz@maths.unine.ch

Vide-leer-embty