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TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

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1. INTRODUCTION

There are a number of ways to define the Teichmüller space of Riemann surfaces. In this paper I treat an approach which is less common than others. Let Γ be a Fuchsian group which uniformizes a closed Riemann surface of genus g. Then a fundamental domain for Γ is chosen in a canonical way, namely as a polygon with $4g$ sides such that opposite sides are identified. The Teichmüller space T_g of closed Riemann surfaces of genus g is then constructed by varying these polygons.

This construction of T_g by polygons was first done by Coldewey and Zieschang in an annex in [17], see also [18]; the construction includes the proof that T_g is homeomorphic to $\mathbf{R}^{6g-6}$. In [2], Buser gave a different, however indirect proof. Here, I propose a new construction and a new proof which is, in my eyes, easier and more transparent than the original one of Coldewey and Zieschang.

The main idea is the following. Let $P(g)$ be a canonical polygon of $4g$ sides which is the fundamental domain of a Fuchsian group uniformizing a closed Riemann surfaces of genus g (the definition of $P(g)$ will include some technical subtleties, to be discussed in Section 3). Then "triangulate" $P(g)$ into $4g-4$ triangles and one quadrilateral S. This can be done in such a way that these triangles are determined by $6g-5$ positive real numbers (corresponding to the lengths of the sides of the triangles) with the condition that the different triangle inequalities hold. It turns out that these $6g-5$ lengths, *taken as homogeneous parameters*, provide a parametrization of the Teichmüller space T_g. Since the set of reals for which the different triangle

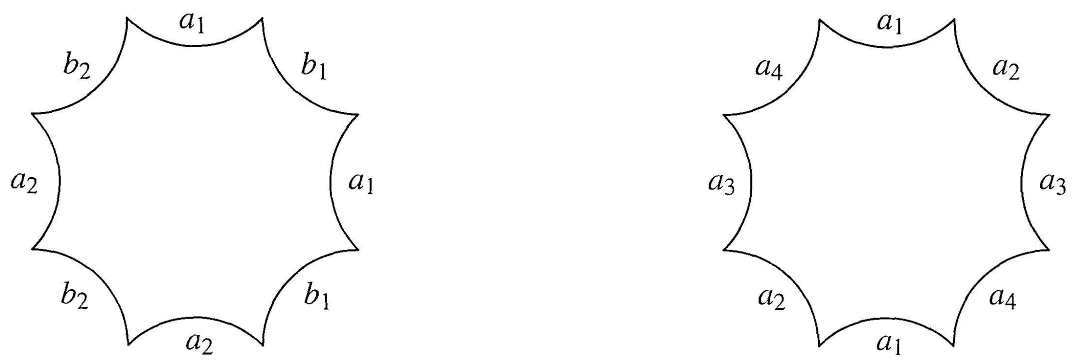

FIGURE 1
On the left hand side: usual identification
On the right hand side: identification chosen in this paper

inequalities hold is open and convex, this also proves that T_g is homeomorphic to $\mathbf{R}^{6g-6}$.

Let P be a polygon of $4g$ sides which is the fundamental domain for a Fuchsian group Γ uniformizing a closed Riemann surface M of genus g. This means that we can write

$$M = \mathbf{H}/\Gamma$$

where $\mathbf{H}$ is the upper halfplane. Usually, P is chosen such that the identification of the sides of P is that of the polygon on the left hand side in Figure 1. The construction described above would equally work for these polygons. For the following reasons I prefer to choose the identification (compare the polygon on the right hand side of Figure 1) such that opposite sides are identified. First the sides of P correspond to simple (this means with no selfintersections) closed curves in M and if opposite sides are identified, then these simple closed curves intersect transversally (which is not the case with the usual identification). Secondly, the vertices of P correspond to a (unique) point Q in M; with the usual identification, Q is completely arbitrary while with the identification chosen here, there is a natural choice for Q in the case of hyperelliptic Riemann surfaces, namely, as a Weierstrass point. See Section 6 for details.

In this paper, I only treat the case of Fuchsian groups which uniformize closed Riemann surfaces. In a straightforward way, the construction and proof could be extended to all finitely generated Fuchsian groups. Note that concerning the original construction and proof in [17] (mentioned above) the corresponding generalization has been worked out by Coldewey in his thesis [3].

The paper is structured as follows. In Section 2 the basic definitions of hyperbolic geometry and Fuchsian groups are given. Section 3 defines the canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space T_g for $g = 2$ can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

2. Hyperbolic geometry and Fuchsian groups

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

DEFINITION. (i) $\mathbf{H} = \{z = (x, y) \in \mathbf{C} : y > 0\}$ denotes the *upper halfplane*. The *hyperbolic metric* on $\mathbf{H}$ is given by

$$dz = \frac{1}{y}(dz)_E$$

where $(dz)_E$ is the standard Euclidean metric on $\mathbf{C}$ and y is the imaginary part of z.

(ii) Define

$$\mathrm{SL}(2,\mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1;\ a, b, c, d \in \mathbf{R} \right\}$$

and

$$\mathrm{PSL}(2,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})/\!\sim$$

with $A \sim B$ if and only if $A = \pm B$ for $A, B \in \mathrm{SL}(2,\mathbf{R})$. Let $\gamma \in \mathrm{SL}(2,\mathbf{R})$,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the action of γ on $\mathbf{H}$ is defined as

$$\gamma(z) = \frac{az+b}{cz+d}$$

for $z \in \mathbf{H}$.

THEOREM 1. $\mathbf{H}$ *is a complete Riemannian manifold of constant curvature* -1. *The geodesics in* $\mathbf{H}$ *are either Euclidean semicircles which are orthogonal to the real axis or vertical half-lines.*

THEOREM 2.

(i) $\mathrm{PSL}(2,\mathbf{R}) = Isom^+(\mathbf{H})$, *the group of orientation preserving isometries of* $\mathbf{H}$.

(ii) *Let* u *and* v *be geodesics in* $\mathbf{H}$, *let* z *be on* u *and* z' *on* v. *Then there exists* $\gamma \in \mathrm{PSL}(2,\mathbf{R})$ *with* $\gamma(u) = v$ *and* $\gamma(z) = z'$.

DEFINITION. For a measurable subset $G \subset \mathbf{H}$ define the *volume* $vol(G)$ as

$$vol(G) = \int_G \frac{dx\,dy}{y^2}.$$

REMARK. The volume is invariant under $\gamma \in \mathrm{SL}(2,\mathbf{R})$.

CONVENTIONS. (i) Speaking of triangles, quadrilaterals and polygons always means that the sides are hyperbolic geodesic segments in $\mathbf{H}$.

(ii) Speaking of *angles* in triangles, quadrilaterals and polygons always means *interior angles*.

THEOREM 3. *The volume of a polygon with angles* α_i, $i = 1, 2, \ldots, m$, $m \geq 3$, *is*

$$(m-2)\pi - \sum_{i=1}^{m} \alpha_i .$$

DEFINITION. A *Fuchsian group* Γ is a *discrete* subgroup of $\mathrm{PSL}(2,\mathbf{R})$ where discrete means that the identity matrix is not a cluster point in Γ with respect to the topology induced by the standard topology of $\mathbf{R}^4$.

THEOREM 4. *Let* Γ *be a Fuchsian group without elliptic elements (an element* $\gamma \in \mathrm{PSL}(2, \mathbf{R})$ *is elliptic if* $|\mathrm{tr}(\gamma)| < 2$ *where* tr *is the trace). Then* $\mathbf{H}/\Gamma$ *is a complete connected orientable Riemannian manifold of dimension* 2 *with a metric of constant curvature* -1.

DEFINITION. A *hyperbolic surface* is a connected orientable manifold $M = \mathbf{H}/\Gamma$ as in Theorem 4 (where Γ is a Fuchsian group without elliptic elements). M is called *closed* if M is compact and has no boundary.

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DEFINITION (Compare Figure 2). Let $g \geq 2$ be an integer. A *canonical polygon* $P(g)$ is a polygon with $4g$ sides, denoted by $a_1, \ldots, a_{4g}$, ordered clockwise, and angles α_i between a_i and a_{i+1}, $i = 1, \ldots, 4g$ (indices are taken modulo $4g$), such that

(I) a_i and a_{i+2g} have the same length, $i = 1, \ldots, 2g$;

(II) the sum of the angles of $P(g)$ is 2π;

(III) $0 < \alpha_i < \pi$, $i = 1, \ldots, 4g$;

(IV) $\alpha_1 = \alpha_{2g+1}$;

(V) $\sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$.

I shall speak of condition (I) (or (II) or (III) or (IV) or (V)) referring to this definition.

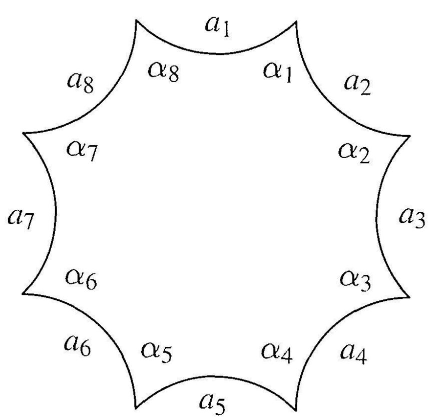

FIGURE 2
A canonical polygon $P(g)$ for $g = 2$

REMARKS. (i) Note that, by condition (II), both sides of the equation in condition (V) equal π.

(ii) The terminology *canonical* polygon is not standard, one finds different objects called canonical polygons in the literature (see for example in [15]).

DEFINITION. Let Γ be a Fuchsian group. A *fundamental domain* for Γ is a measurable subset D of $\mathbf{H}$ such that

(i) $\bigcup_{\gamma\in\Gamma}\gamma(D)=\mathbf{H}$, and

(ii) $int\left(\bar{D}\right)\cap int\left(\gamma\left(\bar{D}\right)\right)=\varnothing$ for $id\neq\gamma\in\Gamma$. Here, $int(S)$ is the *interior* of a set S and id is the unit matrix.

THEOREM 5 (Poincaré). *A canonical polygon $P=P(g)$ is the fundamental domain of a Fuchsian group Γ and $\mathbf{H}/\Gamma$ is a closed hyperbolic surface of genus g. The group Γ is generated by the $2g$ elements γ_i where γ_i is defined by the conditions $\gamma_i(P)\cap int(P)=\varnothing$ and $\gamma_i(a_i)=a_{i+2g}$ if i is odd and $\gamma_i(a_{i+2g})=a_i$ if i is even, $i=1,\dots,2g$.*

REMARKS. (i) For a proof see for example Poincaré [10], Siegel [15], Beardon [1], Iversen [5]. The theorem holds for much more general polygons. A general proof was first given by Maskit [9] and by de Rham [11].

(ii) Traditionally, the $2g$ generators γ_i of a Fuchsian group corresponding to a closed hyperbolic surface of genus g are chosen such that the relation

$$\prod_{i=1}^{2g}\left[\gamma_{2i-1},\gamma_{2i}\right]=id$$

holds where

$$\left[\gamma_{2i-1},\gamma_{2i}\right]=\gamma_{2i-1}\gamma_{2i}(\gamma_{2i-1})^{-1}(\gamma_{2i})^{-1}.$$

With the choice made here, the relation

$$\gamma_1\gamma_2\dots\gamma_{2g}(\gamma_1)^{-1}(\gamma_2)^{-1}\dots(\gamma_{2g})^{-1}=id$$

holds. Compare the introduction for the reasons for this choice.

(iii) Let $P(g)$ be a canonical polygon and $M=\mathbf{H}/\Gamma$ be the corresponding closed hyperbolic surface. Then the vertices of $P(g)$ correspond to a unique point Q in M and the side a_i (as well as a_{2g+i}) of $P(g)$ corresponds to a simple closed curve u_i in M, $i=1,\dots,2g$. These curves all intersect transversally in Q and intersect in no other point. Moreover, these curves are geodesic loops based in Q, this means that the curves may have an angle $\neq\pi$ in Q, but outside Q, they are geodesic. Further, condition (IV) and condition (V) of canonical polygons are equivalent to the condition that u_1 and u_2 are simple closed geodesics in M.

4. TRIGONOMETRY

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

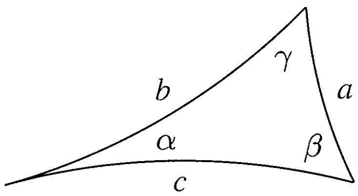

FIGURE 3
The notation for a triangle

THEOREM 6. *Let T be a triangle with angles α, β, γ and sides of length a, b, c with the the notation of Figure 3. Then*

(i) $\dfrac{\sinh a}{\sin \alpha} = \dfrac{\sinh b}{\sin \beta} = \dfrac{\sinh c}{\sin \gamma}$;

(ii) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$;

(iii) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$.

LEMMA 7. *Let T be a triangle with the notation of Figure 3. Let T' be a triangle with sides of length a', b', c' and angles α', β', γ'. Let $a = a'$ and $b = b'$. Then*

$$c' > c \Longleftrightarrow \gamma' > \gamma \Longleftrightarrow \alpha' + \beta' < \alpha + \beta .$$

Proof. The first equivalence is a consequence of Theorem 6 (ii).

Let Z be the centre of the side c and let u be the geodesic segment, of length $d/2$ say, between Z and the vertex C of T. The segment u separates T into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$

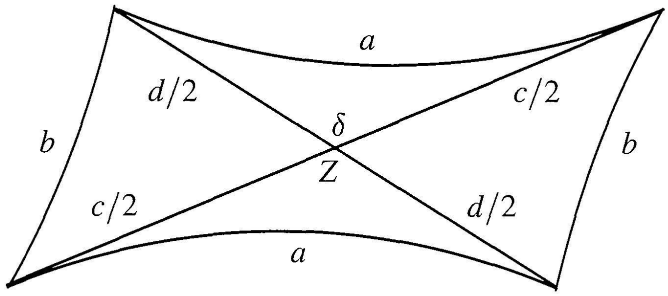

FIGURE 4
The triangle T (thick lines) is half of this quadrilateral

and

$$\cosh b = \cosh(c/2)\cosh(d/2) + \sinh(c/2)\sinh(d/2)\cos\delta$$

for an angle δ. This implies

$$\cosh a + \cosh b = 2\cosh(c/2)\cosh(d/2)\,. \tag{1}$$

Let $\widetilde{T}$ be the triangle with sides of length a, b, d (compare Figure 4). Then the angles of $\widetilde{T}$ are $\alpha+\beta, \gamma_1, \gamma_2$ with $\gamma = \gamma_1 + \gamma_2$. Now if the length of c grows, then the length of d diminishes (by (1)), therefore, applying the first equivalence of the lemma to the triangle $\widetilde{T}$, the angle $\alpha+\beta$ diminishes and the second equivalence of the lemma follows. □

COROLLARY 8. *Let Q and Q' be two quadrilaterals with the same lengths of the four sides. Let $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ be the four angles in Q and Q', respectively, in the natural order (α and γ are opposite). Then*

$$\alpha + \gamma > \alpha' + \gamma' \iff \beta + \delta < \beta' + \delta'\,.$$

Proof. Clear by Lemma 7 (draw a diagonal in Q and in Q'). □

LEMMA 9. *Let T be a triangle with the notation of Figure 3. Let $T(t)$ be a triangle with sides of length ta, tb, tc and angles $\alpha_t, \beta_t, \gamma_t$.*

(i) *If $t > 1$, then $\alpha_t < \alpha$, $\beta_t < \beta$, $\gamma_t < \gamma$.*

(ii) *For $t \to \infty$, the three angles $\alpha_t, \beta_t, \gamma_t$ converge to zero.*

Proof. (i) I prove $\gamma_t < \gamma$, the two other inequalities follow analogously. By Theorem 6 (ii) it has to be shown that

$$\frac{\cosh ta \cosh tb - \cosh tc}{\sinh ta \sinh tb} - \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b} > 0\,. \tag{2}$$

By symmetry we can assume that $a \geq b$. Consider the left hand side of (2) as a function $f = f(c)$ of c with fixed a, b, t. A calculation yields

(3) $$f(a+b) = f(a-b) = 0.$$

Further, $f'(c) = 0$ implies

$$\frac{t \sinh tc}{\sinh c} = \frac{\sinh ta \sinh tb}{\sinh a \sinh b}$$

and by the convexity of the function sinh we conclude that $f'(c)$ has only one zero. Since $t > 1$, it follows (by the definition of f) that

$$f(c) \to -\infty \text{ for } c \to \pm\infty.$$

Therefore, by (3), $f(c) > 0$ for $a - b < c < a + b$, which is the triangle inequality, and $\gamma_t < \gamma$ follows.

(ii) Assume without restriction that $a \leq b \leq c$. It then follows by Theorem 6 (i) that $\alpha \leq \beta \leq \gamma$. This implies by Theorem 6 (iii) that α_t and β_t converge to zero for $t \to \infty$. We compare the triangle $T(t)$ with the triangle $T'(t)$ which has two sides of length $t(a+b)/2$ and one side of length tc. Denote by γ_t' the angle in $T'(t)$ which is opposite to the side of length tc. By a similar (but easier) argument as in part (i) it follows that $\gamma_t' \geq \gamma_t$ for all $t \geq 1$. It is therefore sufficient to prove

(4) $$\gamma_t' \to 0, \quad \text{for } t \to \infty.$$

By Theorem 6 (i) we have

$$\sin \frac{\gamma_t'}{2} = \frac{\sinh(tc/2)}{\sinh(t(a+b)/2)}.$$

This implies (4) since $c/2 < (a+b)/2$ (by the triangle inequality). □

COROLLARY 10. *Let Q be a quadrilateral with sides of length a, b, c, d and angles $\alpha, \beta, \gamma, \delta$ (so that a and c are opposite sides and α and γ are opposite angles). Let $Q(t)$ be a quadrilateral with sides of length ta, tb, tc, td and angles $\alpha_t, \beta_t, \gamma_t, \delta_t$ (the notation is analogous to that of Q).*

(i) *If $t > 1$, then at least two opposite angles are smaller in $Q(t)$ than in Q.*

(ii) *For every $\epsilon > 0$, there exists a real $T(\epsilon)$ such that, for every $t > T(\epsilon)$, $\alpha_t + \gamma_t < \epsilon$ or $\beta_t + \delta_t < \epsilon$.*

Proof. Let e be the length of a diagonal of Q. Construct the quadrilateral $Q'(t)$ with a diagonal of length te and sides of length ta, tb, tc, td. By Lemma 9 all four angles of $Q'(t)$ are smaller than the corresponding angles in Q and moreover converge to zero if $t \to \infty$. The corollary now follows by Corollary 8. □

5. Teichmüller space

DEFINITION. The *space $\mathcal{P}(g)$ of canonical polygons* contains all canonical polygons $P(g)$ with the topology $P_j(g) \to P(g)$ if and only if the lengths of all sides converge and all angles converge, more precisely, if and only if

$$a_i(P_j(g)) \to a_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $a_i(P_j(g))$ is the side a_i of $P_j(g)$) and

$$\alpha_i(P_j(g)) \to \alpha_i(P(g)), \quad i = 1, \dots, 4g,$$

(where $\alpha_i(P_j(g))$ is the angle α_i of $P_j(g)$).

REMARKS. (i) Note that two canonical polygons $P(g)$ and $P'(g)$ may be isometric, but represent different points in $\mathcal{P}(g)$. They represent the same point if and only if there is an isometry mapping the side $a_i(P(g))$ to the side $a_i(P'(g))$, $i = 1, \dots, 4g$ (and not to the side $a_j(P'(g))$, $j \neq i$). One expresses this fact by saying that the sides of the canonical polygons are *marked.*

(ii) One may calculate the dimension of $\mathcal{P}(g)$ in the following heuristic way (this argument is modeled after one given in [16]). A canonical polygon has $4g$ vertices. Each vertex is determined in $\mathbf{H}$ by two (real) parameters, this gives $8g$ parameters. The dimension of the space of isometries of $\mathbf{H}$ is 3 so we remain with $8g - 3$ parameters. By condition (I) of a canonical polygon we have $2g$ equalities and each of the conditions (II), (IV), (V) gives one equality. We remain with

$$8g - 3 - 2g - 3 = 6g - 6$$

parameters.

THEOREM 11. *$\mathcal{P}(g)$ is homeomorphic to $\mathbf{R}^{6g-6}$.*

REMARK. The following proof is new. The theorem was first proved by Coldewey and Zieschang in an annex to [17], see also [18]. An (indirect) proof has also been given by Buser [2], compare the introduction.

Proof. (i) Let $P(g)$ be a canonical polygon with sides a_i and angles α_i between a_i and a_{i+1}, $i = 1, \dots, 4g$ (the indices are taken modulo $4g$). Let $\{Q_i\} = a_i \cap a_{i+1}$, $i = 1, \dots, 4g$. Denote by b_i the geodesic segment between

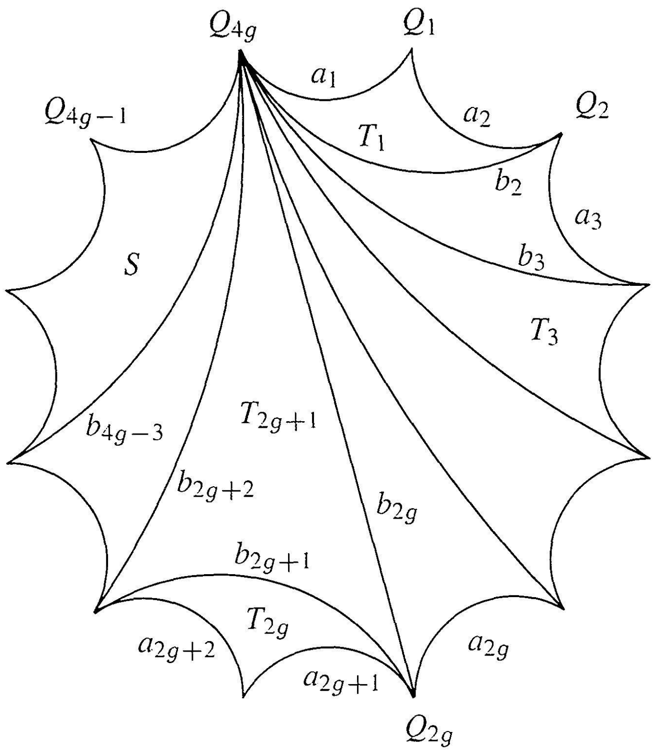

FIGURE 5

The "triangulation" of a canonical polygon $P(g)$

Q_{4g} and Q_i, $i = 2, \ldots, 4g-3$, $i \neq 2g+1$. Denote by b_{2g+1} the geodesic segment between Q_{2g} and Q_{2g+2}, compare Figure 5.

$P(g)$ is separated by the geodesic segments $b_2, \ldots, b_{4g-3}$ into one quadrilateral S and $4g-4$ triangles T_i, $i = 1, \ldots, 4g-4$, with sides b_i, b_{i+1}, a_{i+1} for $i = 2, \ldots, 4g-4$, $i \neq 2g$, $i \neq 2g+1$; the triangle T_1 has sides a_1, a_2, b_2, the triangle T_{2g} has sides $a_{2g+1}, a_{2g+2}, b_{2g+1}$, and the triangle T_{2g+1} has sides $b_{2g}, b_{2g+1}, b_{2g+2}$ (note that T_{2g+1} is only defined if $g > 2$).

A point $x = (x_1, \ldots, x_{6g-5}) \in \mathbf{R}^{6g-5}$ is called *admissible* if $x_j > 0$, $j = 1, \ldots, 6g-5$, and if, putting

$$L(a_i) = L(a_{i+2g}) = x_i, \quad i = 1, \ldots, 2g \quad (L = \text{length})$$

and

$$L(b_2) = L(b_{2g+1}) = x_{2g+1}$$

and

$$L(b_i) = x_{2g+i-1}, \quad i = 3, \ldots, 2g; \quad L(b_i) = x_{2g+i-2}, \quad i = 2g+2, \ldots, 4g-3,$$

the triangle inequalities hold for the triangles T_k, $k = 1, \ldots, 4g-4$, and the "quadrilateral inequalities" hold for S (which means that the sum of the lengths of any three sides of S is greater than the length of the fourth side). Note that these are purely algebraic conditions on $x \in \mathbf{R}^{6g-5}$.

Let O be the subset of $\mathbf{R}^{6g-5}$ of admissible points. Being the intersection of a finite number of open sets, O is open. Moreover, O is convex since O is the intersection of a finite number of convex sets, namely, if for example $x_1 + x_2 > x_3$ and $y_1 + y_2 > y_3$, then

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) > \lambda x_3 + (1 - \lambda)(y_3), \ \forall \lambda \in [0, 1] .$$

(ii) Let $x \in O$. Then we associate a formal polygon $P(x)$ to x in the following way. $P(x)$ is the formal union of the triangles $T_k(x)$, $k = 1, \dots, 4g - 4$, and the quadrilateral $S(x)$ in the same way as $P(g)$. Hereby, the triangles, as well as the lengths of the sides of $S(x)$ are defined by the identifications described in part (i). The angles of the triangles are determined by their sides (by Theorem 6). The (formal) angles α_i of $P(x)$, $i = 1, \dots, 4g$, are defined as the sum of the angles of the corresponding triangles and (if $i \in \{4g - 3, 4g - 2, 4g - 1, 4g\}$) of $S(x)$. Thereby, the angles of $S(x)$ are defined by the conditions that $S(x)$ is convex and that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal, this minimum is denoted by $\mathbf{m}(x)$. By Corollary 10 the angles of $S(x)$ are then determined and hence also the angles of $P(x)$. Note however that an angle α_i of $P(x)$ may be greater than 2π, this is why $P(x)$ is called a formal polygon with formally defined angles.

(iii) Let $x \in O$. Then tx (for $t \in \mathbf{R}$, $t > 0$) is also in O (since the triangle inequalities remain true). I claim that there exists a unique $t_0 > 0$ (depending on x) such that $P(t_0 x)$ is a canonical polygon. I first show uniqueness. Assume that $\mathbf{m}(tx) > 0$ for $P(tx)$. This means that $\mathbf{A}(tx) - \mathbf{B}(tx) \neq 0$ where

$$\mathbf{A}(tx) := \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} \quad \text{and} \quad \mathbf{B}(tx) := \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1} .$$

If $\mathbf{A}(tx) - \mathbf{B}(tx) > 0$, then an angle in $S(tx)$ must be π and, by Corollary 8 and the minimality of $\mathbf{m}(x)$, this angle must appear in the sum $\mathbf{B}(tx)$. This implies that

$$\Sigma(tx) := \mathbf{A}(tx) + \mathbf{B}(tx) > 2\pi . \tag{5}$$

Of course, (5) also holds if $\mathbf{A}(tx) - \mathbf{B}(tx) < 0$. It follows that if $P(t_0 x)$ is a canonical polygon, then $\mathbf{m}(t_0 x) = 0$ (since $\Sigma(t_0 x) = 2\pi$ by the definition of canonical polygons). Now assume that $P(t_0 x)$ and $P(t_1 x)$ are canonical polygons with $t_1 > t_0$. By Lemma 9, all angles of the triangles $T_k(t_1 x)$

are smaller than the corresponding angles in $T_k(t_0x)$, $k = 1, \ldots, 4g-4$. Moreover, by Corollary 10, at least two opposite angles in $S(t_1x)$ are smaller than the corresponding angles in $S(t_0x)$. This implies that $\mathbf{A}(t_1x) < \mathbf{A}(t_0x)$ or $\mathbf{B}(t_1x) < \mathbf{B}(t_0x)$. But since $\mathbf{A}(t_1x) = \mathbf{B}(t_1x)$ and $\mathbf{A}(t_0x) = \mathbf{B}(t_0x)$ ($\mathbf{m}(t_0x) = \mathbf{m}(t_1x) = 0$), it follows that $\Sigma(t_1x) < \Sigma(t_0x)$, a contradiction. This proves uniqueness.

As for existence note that if $t \to 0$, then the volume of all triangles T_k, $k = 1, \ldots, 4g-4$, and the volume of S tend to zero which implies by Theorem 3 that

$$\Sigma := \sum_{i=1}^{4g} \alpha_i \to (4g-2)\pi .$$

On the other hand, for $t \to \infty$, all angles in the triangles T_k, $k = 1, \ldots, 4g-4$, converge to zero by Lemma 9 and, by Corollary 10 (ii), at least two opposite angles of S converge to zero. It follows by the condition that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal that all angles of S converge to zero and hence Σ converge to zero. Therefore, there exists a t_0 such that $\Sigma(t_0x) = 2\pi$. Now $P(t_0x)$ is a canonical polygon. Namely, conditions (I), (II) and (IV) hold by construction. By the argument above, we further have $\mathbf{m}(t_0x) = 0$ and condition (V) holds. Finally, condition (III) holds since all sides of the triangles of $P(t_0x)$ have finite length and since conditions (II) and (V) hold.

(iv) We therefore have defined a projection from the open convex set O to the unit sphere in $\mathbf{R}^{6g-5}$. Since all operations are controlled by the formulas of Theorem 6, it is clear that this map is continuous and that the image is homeomorphic to $\mathbf{R}^{6g-6}$ as well as homeomorphic to $\mathcal{P}(g)$ since every canonical polygon is thereby obtained. □

DEFINITION. By Theorem 5 each of the canonical polygons in $\mathcal{P}(g)$ defines a closed hyperbolic surface of genus g. The *Teichmüller space* T_g is the space of these hyperbolic surfaces with the topology induced from that of $\mathcal{P}(g)$.

COROLLARY 12. *T_g is homeomorphic to* $\mathbf{R}^{6g-6}$. □

6. Applications

LEMMA 13. *Let M be a closed hyperbolic surface of genus g which has $2g-2$ simple closed geodesics $u_1, \dots, u_{2g-2}$ which all intersect in the same point Q and intersect in no other point. Then M has simple closed curves u_{2g-1} and u_{2g}, passing through Q, such that the curves u_i intersect in no other point than Q, $i = 1, \dots, 2g$. Moreover, u_{2g-1} and u_g can be chosen such that*

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon $P(g)$.

Proof. Cut M along u_1, the result is a hyperbolic surface M_1 with boundary and genus $g-1$, the boundary consists of two simple closed geodesics v_1 and w_1. Cut M_1 along u_2, the result is a hyperbolic surface M_2 with one boundary component v_2 and genus $g-1$. Now cut M along all $2g-2$ simple closed geodesics $u_1, \dots, u_{2g-2}$. By induction, the result is a hyperbolic surface M_{2g-2} with one boundary component v and genus 1. More precisely, the boundary v is piecewise geodesic with $4g-4$ pieces and we may assume that the notation is chosen such that these pieces appear on v in the order (the pieces are called like the corresponding closed curves) $u_1, u_2, \dots, u_{2g-2}, u_1, u_2, \dots, u_{2g-2}$ (note that closed geodesics intersect transversally). Denote by S and S' the two copies of Q on v between u_1 and u_{2g-2}. Let u_{2g-1} be a simple geodesic in M_{2g-2} which joins S and S' such that u_{2g-1} is not homotopic to a part of v. Cut M_{2g-2} along u_{2g-1}. The result is a hyperbolic surface M_{2g-1} of genus zero with two boundary components w and w' which both consist of $2g-1$ geodesic pieces in the order $u_1, u_2, \dots, u_{2g-2}, u_{2g-1}$. Denote by R and R' the copies of Q between u_1 and u_{2g-1} on w and w', respectively. Let u_{2g} be a simple geodesic in M_{2g-1} which joins R and R', u_{2g} can be chosen such that when we cut M_{2g-1} along u_{2g}, then we obtain the interior of a canonical polygon as desired. □

DEFINITION. A *hyperelliptic surface* is a closed hyperbolic surface of genus g which has an isometry ϕ with $\phi^2 = id$ and with exactly $2g+2$ fixed points.

In [14], the equivalence of (i) and (ii) of the following theorem was first proved. With the approach chosen here, we can give a third equivalence and a different proof.

THEOREM 14. *Let M be a closed hyperbolic surface M of genus g. Then the following conditions are equivalent.*

(i) *M is hyperelliptic.*

(ii) *M has a set of at least $2g-2$ simple closed geodesics which all intersect in the same point and intersect in no other point.*

(iii) *M has a corresponding canonical polygon with equal opposite angles ($\alpha_i = \alpha_{2g+i}$, $i = 1, \dots, 2g$).*

Proof. I shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Let M be hyperelliptic. Let R_i, $i = 1, \dots, 2g+2$, be the fixed points of a hyperelliptic involution ϕ. Let c_1 be a simple geodesic segment from R_1 to R_2. Then $c_1 \cup \phi(c_1)$ is a simple closed geodesic u_1 since $\phi^2 = id$. It also follows that on u_1, there are only two fixed points of ϕ and that $M_1 = M \setminus u_1$ is connected. Therefore, we can choose a simple geodesic segment c_2 from R_1 to R_3 which intersects u_1 only in R_1. By the same argument as above, $c_2 \cup \phi(c_2)$ is a simple closed geodesic, $M_2 = M \setminus (u_1 \cup u_2)$ is connected and on $u_1 \cup u_2$, there are only three fixed points of ϕ. Continuing this construction we can find simple closed geodesics $u_1, \dots, u_{2g-2}$ which all intersect in R_1 and in no other point. This proves (i) $\Rightarrow$ (ii).

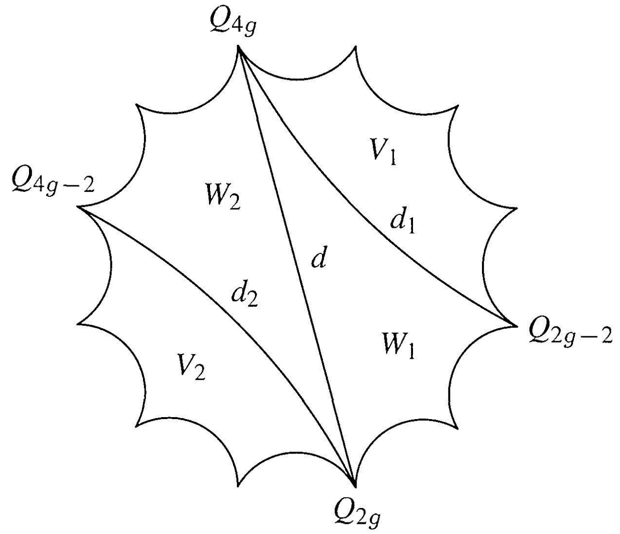

FIGURE 6
The partition of a canonical polygon $P(g)$ into two $(2g-1)$-gons and two quadrilaterals

Assume now that M has $2g-2$ simple closed geodesics $u_1, \dots, u_{2g-2}$ which all intersect in the same point Q and intersect in no other point. By Lemma 13 we then can find simple closed curves u_{2g-1} and u_{2g} such that

$$M \setminus \bigcup_{i=1}^{2g} u_i$$

is the interior of a canonical polygon $P(g)$ with the usual notation. For $i = 1, \ldots, 4g$, let $\{Q_i\} = a_i \cap a_{i+1}$. In $P(g)$ let d_1 be the geodesic segment from Q_{4g} to Q_{2g-2}, d_2 the geodesic segment from Q_{2g} to Q_{4g-2}, and d the geodesic segment from Q_{2g} to Q_{4g}, compare Figure 6. Then $P(g) \setminus (d_1 \cup d_2 \cup d)$ has four connected components, two quadrilaterals W_j having d and d_j, $j = 1, 2$, among the sides and two $(2g-1)$-gons V_j having d_j among the sides, $j = 1, 2$. Since u_i, $i = 1, \ldots, 2g-2$, are simple closed geodesics, it follows that $\alpha_i = \alpha_{i+2g}$ for $i = 1, \ldots, 2g-3$. This implies that V_1 and V_2 are isometric and that d_1 and d_2 have the same length. Therefore, W_1 and W_2 are quadrilaterals with equal lengths of the four sides. Fix now W_1 and try to vary W_2 such that the lengths of the sides remain invariant and so that property (V) for canonical polygons holds. This is certainly the case if W_2 and W_1 are isometric. But then Corollary 8 implies that this is the unique possibility. Therefore, W_1 and W_2 must be isometric and hence $\alpha_i = \alpha_{i+2g}$ for all $i = 1, \ldots, 2g$, which proves (ii) $\Rightarrow$ (iii).

Now assume that (iii) holds. Let d be the geodesic segment from Q_{2g} to Q_{4g}. Then d separates $P(g)$ into two isometric $(2g+1)$-gons and the π-rotation around the centre C of d induces an isometry ϕ of M with $\phi^2 = id$. The fixed points of ϕ are C, the point Q corresponding to the vertices of $P(g)$ as well as the centres of the sides a_i, $i = 1, \ldots, 2g$. Therefore, ϕ is a hyperelliptic involution which proves (iii) $\Rightarrow$ (i). □

COROLLARY 15. *All closed hyperbolic surfaces of genus 2 are hyperelliptic.*

Proof. All closed hyperbolic surfaces have two simple closed geodesics which intersect in a unique point. The corollary follows by Theorem 14. □

DEFINITION. Let M_0 be a closed hyperbolic surface in T_g. For every $M \in T_g$ fix a homeomorphism ϕ_M, homotopic to the identity, from M_0 to M (ϕ_M exists since closed surfaces of the same genus are homeomorphic). Let u be a simple closed geodesic in M_0. Then, in the homotopy class of $\phi_M(u)$ there exists a unique simple closed geodesic which is denoted by $\Phi_M(u)$. The function

$$L(u)\colon T_g \to \mathbf{R}$$

which associates to M the length of $\Phi_M(u)$ is called a *geodesic length function*.

REMARK. It is well known that T_g can be parametrized by a finite number of geodesic length functions, see for example [12], [13] where it is shown that T_g can be parametrized by $6g-5$ geodesic length functions.

THEOREM 16. *The Teichmüller space T_g for $g=2$ can be parametrized by 7 (suitably chosen) geodesic length functions $L(u_1),\dots,L(u_7)$, taken as homogeneous parameters (which means that $L(u_1)/L(u_7),\dots,L(u_6)/L(u_7)$ gives a parametrization of T_2).*

Proof. Let $P(2)$ be a canonical polygon corresponding to a closed hyperbolic surface M_0 of genus 2. As usual let $Q_i = a_i \cap a_{i+1}$, $i=1,\dots,8$, where the a_i are the sides of $P(2)$. Let b_i be the geodesic segment (in $P(2)$) between Q_i and Q_{i+4}, $i=1,\dots,4$. By Corollary 15, M_0 is hyperelliptic, therefore (compare Theorem 14) b_i corresponds to a simple closed geodesic in M_0, denoted by B_i, $i=1,\dots,4$. It also follows by Theorem 14 that a_i corresponds to a simple closed geodesic in M_0, denoted by A_i, $i=1,\dots,4$.

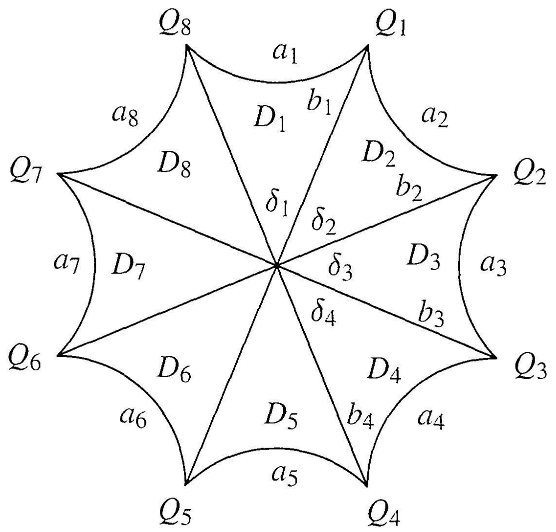

FIGURE 7

A triangulation of a canonical polygon $P(g)$ for $g=2$

I now prove that the 7 length functions, given by the simple closed geodesics A_i, $i=1,2,3$, B_i, $i=1,\dots,4$, taken as homogeneous parameters, give a parametrization of T_2. In order to do this, it is enough (by Theorem 11 and Corollary 12) to show that $P(2)$ is uniquely determined by the lengths of a_i, $i=1,2,3$, b_i, $i=1,\dots,4$, taken as homogeneous parameters (in the sequel I shall refer to these lengths calling them "the seven lengths"). This can be done analogously as in the proof of Theorem 11. The geodesic segments b_i, $i=1,\dots,4$, intersect in a point C, the "centre" of $P(2)$, and they separate

$P(2)$ into 8 triangles D_j so that a_j is a side of D_j, $j = 1, \ldots, 8$, compare Figure 7. Since M is hyperelliptic, D_j and D_{j+4} are isometric, $j = 1, \ldots, 4$. Denote by δ_i the angle of D_i in the vertex C, $i = 1, \ldots, 4$. The seven lengths determine the triangles D_i, $i = 1, 2, 3$, as well as two sides and the angle δ_4 of D_4 by the condition

$$\Delta := \sum_{j=1}^{4} \delta_j = \pi , \tag{6}$$

so they determine also D_4. This shows that the seven lengths determine $P(2)$. Multiply the seven lengths by a positive real t and assume that the seven new lengths also determine a canonical polygon $P_t(2)$. If $t > 1$, then δ_i, $i = 1, 2, 3$, are smaller in $P_t(2)$ than in $P(2)$ by Lemma 9, therefore, by (6), δ_4 is larger in $P_t(2)$ than in $P(2)$. It follows by Lemma 7 that the sum of the two other angles of D_4 is smaller in $P_t(2)$ than in $P(2)$. Since all angles in D_i, $i = 1, 2, 3$, are smaller in $P_t(2)$ than in $P(2)$ by Lemma 9, it follows that

$$\sum_{i=1}^{4} \alpha_i$$

is smaller in $P_t(2)$ than in $P(2)$. But this contradicts condition (II) of canonical polygons. An analogous contradiction follows if $t < 1$ proving thus that $t = 1$ and therefore the theorem. □

REMARK. Theorem 16 is new. It is well known that $6g-6$ length functions can never parametrize T_g so that the situation of Theorem 16 is the best we can expect. It is not known whether $6g-5$ geodesic length functions, *taken as homogeneous parameters*, can parametrize T_g for $g \geq 3$.

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